Calculus

Techniques of Integration

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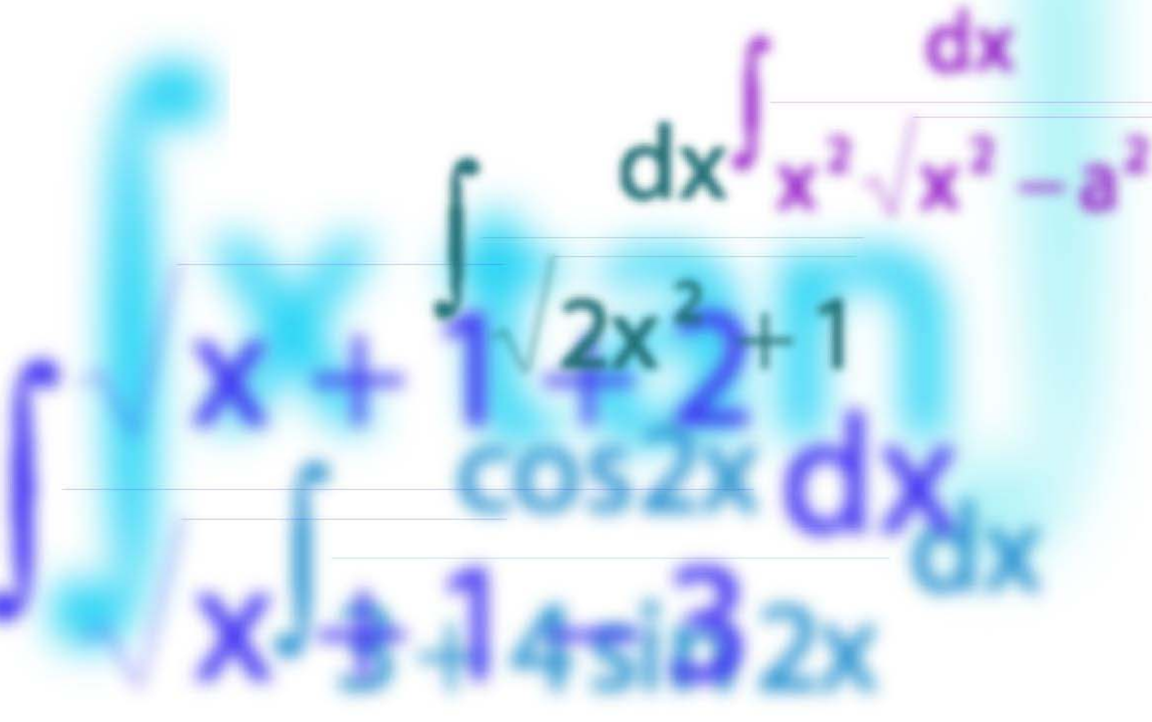
- ✓ Integration formulas
- ✓ 300 solved problems
- ✓ Quick search
- ✓ The ideal guide for self-study

$$\int_0^{\ln 2} xe^{-x}dx$$



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Preface

This book gives a straightforward introduction to techniques of integration, which is one of the most difficult areas of calculus. It is well known that the only way to learn calculus is by solving problems. About 300 completely worked examples are used to introduce methods of integration and to demonstrate problem-solving techniques. The sections are written as self-guided tutorials. Each chapter opens with appropriate definitions and formulas followed by a lot of solved problems listed in order of increasing difficulty. In addition, Appendices at the end of the book include reduction formulas, trigonometric and hyperbolic formulas and identities.

Look carefully at the worked out examples and go through them step by step, then try to solve similar problems. The more problems you work, the better you become at solving them. You can also use the material as a reference source, or as a "database" of solved problems.

Studying and solving these problems improves understanding of techniques of integration and helps students to be ready for test works and exams in the fastest, most efficient manner available.

Chapter 12

Improper Integral

The definite integral $\int_a^b f(x)dx$ is called an improper integral if one of two situations occurs:

- a or b is infinite
- f(x) has one or more points of discontinuity in the interval [a, b].

Infinite Limits of Integration

Let f(x) be a continuous function on the closed interval $[a,\infty)$. We define the improper integral as

$$\int_a^{\infty} f(x)dx = \lim_{n\to\infty}\int_a^n f(x)dx.$$

Let f(x) be a continuous function on the closed interval $(-\infty, b]$. In this case, we define the improper integral as

$$\int_{-\infty}^{b} f(x)dx = \lim_{n\to-\infty}\int_n^b f(x)dx.$$

If these limits exist and are finite then we say that the integrals are convergent; otherwise the integrals are divergent.

Let f(x) be a continuous function for all real numbers. We define

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{\infty} f(x)dx .$$

If, for some real number c, both of the integrals in the right side are convergent, then we say that the integral $\int_{-\infty}^{\infty} f(x)dx$ is also convergent; otherwise it is divergent.

Comparison Theorems

Let f(x) and g(x) be continuous functions on the closed interval $[a,\infty)$. Suppose that $0 \le g(x) \le f(x)$ for all x in the interval $[a,\infty)$.

1. If $\int_{a}^{\infty} f(x)dx$ is convergent then $\int_{a}^{\infty} g(x)dx$ is also convergent;
2. If $\int_{a}^{\infty} g(x)dx$ is divergent then $\int_{a}^{\infty} f(x)dx$ is also divergent.
3. If $\int_{a}^{\infty} |f(x)|dx$ is convergent then $\int_{a}^{\infty} f(x)dx$ is also convergent. In this case, we say that the integral $\int_{a}^{\infty} f(x)dx$ is absolutely convergent.

Discontinuous Integrand

Let f(x) be a function which is continuous on the interval [a, b) but is discontinuous at x = b. We define

$$\int_{a}^{b} f(x)dx = \lim_{\varepsilon \to 0+} \int_{a}^{b-\varepsilon} f(x)dx .$$

Let f(x) be a function which is continuous on the interval (a, b] but is discontinuous at x = a. We define

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b f(x)dx .$$

If these limits exist and are finite then we say that the integrals are convergent; otherwise the integrals are divergent.

Let f(x) be a continuous function for all real numbers x in the interval [a, b] except for some point c in (a, b). We define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

and say that the integral $\int_a^b f(x)dx$ is convergent if both of the integrals in the right side are also convergent; otherwise it is divergent.

SOLVED PROBLEMS

Example 186.

Evaluate $\int_1^\infty \frac{dx}{\sqrt{x}}$.

Solution.

By the definition of an improper integral, we have

$$\int_1^\infty \frac{dx}{\sqrt{x}} = \lim_{n\to\infty}\int_1^n \frac{dx}{\sqrt{x}} = \lim_{n\to\infty}\int_1^n x^{-\frac{1}{2}}dx = \lim_{n\to\infty}\left(\frac{x^{-\frac{1}{2}+1}}{1/2}\right)\Bigg|_1^n$$

$$=\lim_{n\to\infty}\left(2x^{\frac{1}{2}}\right)\Bigg|_1^n=\lim_{n\to\infty}\left(2\sqrt{x}\right)\Big|_1^n=2\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{1}\right)=\infty .$$

Hence, the given integral diverges.

Example 187.

Determine for what values of k the integral $\int_1^\infty \frac{dx}{x^k}$ $(k>0,\ k\neq 1)$ converges.

Solution.

By the definition of an improper integral, we have

$$\int_1^\infty \frac{dx}{x^k}=\lim_{n\to\infty}\int_1^n \frac{dx}{x^k}=\lim_{n\to\infty}\int_1^n x^{-k}dx=\lim_{n\to\infty}\left(\frac{x^{-k+1}}{-k+1}\right)\Bigg|_1^n=\frac{1}{1-k}\cdot\lim_{n\to\infty}\left(x^{-k+1}\right)\Big|_1^n$$

$$=\frac{1}{1-k}\cdot\lim_{n\to\infty}\left(n^{-k+1}-1^{-k+1}\right)=\frac{1}{k-1}\cdot\lim_{n\to\infty}\left(1-n^{1-k}\right).$$

As seen from the expression, there are 2 cases:

- If $0<k<1$, then $n^{1-k}\to\infty$ as $n\to\infty$ and the integral diverges;
- If $k>1$, then $n^{1-k}=\frac{1}{n^{k-1}}\to 0$ as $n\to\infty$ and the integral converges.

Example 188.

Evaluate $\int_0^\infty \frac{dx}{x^2+16}$.

Solution.

$$\int_0^{\infty}\frac{dx}{x^2+16}=\lim_{n\to\infty}\int_0^{n}\frac{dx}{x^2+16}=\lim_{n\to\infty}\int_0^{n}\frac{dx}{x^2+4^2}=\lim_{n\to\infty}\left(\frac{1}{4}\tan^{-1}\frac{x}{4}\right)\Bigg|_0^n$$

$$=\frac{1}{4}\lim_{n\to\infty}\left(\tan^{-1}\frac{n}{4}-\tan^{-1}\frac{0}{4}\right)=\frac{1}{4}\lim_{n\to\infty}\left(\tan^{-1}\frac{n}{4}-0\right)=\frac{1}{4}\cdot\frac{\pi}{2}=\frac{\pi}{8}.$$

Hence, the given integral converges.

Example 189.

Evaluate $\int_{-\infty}^{0}3^x\,dx$.

Solution.

By the definition of an improper integral, we have

$$\int_{-\infty}^{0}3^x\,dx=\lim_{n\to\infty}\int_{-n}^{0}3^x\,dx=\lim_{n\to\infty}\left(\frac{3^x}{\ln 3}\right)\Bigg|_{-n}^{0}=\frac{1}{\ln 3}\cdot\lim_{n\to\infty}\left(3^0-3^{-n}\right)$$

$$=\frac{1}{\ln 3}\cdot\lim_{n\to\infty}\left(1-\frac{1}{3^n}\right)=\frac{1}{\ln 3}\cdot(1-0)=\frac{1}{\ln 3}.$$

Thus, the integral converges.

Example 190.

Evaluate $\int_0^{\infty}e^{-3x}dx$.

Solution.

By the definition of an improper integral, we have

$$\int_0^{\infty}e^{-3x}dx=\lim_{n\to\infty}\int_0^{n}e^{-3x}dx=\lim_{n\to\infty}\left(\frac{e^{-3x}}{-3}\right)\Bigg|_0^n$$

$$= -\frac{1}{3}\lim_{n\to\infty}\left(e^{-3n} - e^{0}\right) = -\frac{1}{3}(0-1) = \frac{1}{3}.$$

Hence, the given integral converges.

Example 191.

Evaluate $\int_0^\infty \frac{xdx}{\left(x^2+3\right)^4}$.

Solution.

By the definition of an improper integral, we have

$$I = \int_0^\infty \frac{xdx}{\left(x^2+3\right)^4} = \lim_{n\to\infty}\int_0^n \frac{xdx}{\left(x^2+3\right)^4}.$$

Make the substitution $x^2+3=t$, $2xdx=dt$, $xdx=\frac{dt}{2}$. Then $t=3$, when $x=0$, and $t=n^2+3$, when $x=n$, so the integral is

$$I = \lim_{n\to\infty}\int_0^n \frac{xdx}{\left(x^2+3\right)^4} = \lim_{n\to\infty}\int_3^{n^2+3}\frac{dt/2}{t^4} = \frac{1}{2}\lim_{n\to\infty}\int_3^{n^2+3} t^{-4}dt$$

$$= \frac{1}{2}\lim_{n\to\infty}\left(\frac{t^{-4+1}}{-4+1}\right)\Bigg|_3^{n^2+3} = -\frac{1}{6}\lim_{n\to\infty}\left(t^{-3}\right)\Big|_3^{n^2+3}$$

$$= -\frac{1}{6}\lim_{n\to\infty}\left(\frac{1}{\left(n^2+3\right)^3} - \frac{1}{3^3}\right) = -\frac{1}{6}\left(0-\frac{1}{27}\right) = \frac{1}{162}.$$

Hence, the given integral converges.

Example 192.

Evaluate the integral $\int_{-2}^{2}\frac{dx}{x^3}$.

Solution.

There is a discontinuity at x = 0 so that we must consider two improper integrals:

$$\int_{-2}^{2}\frac{dx}{x^3}=\int_{-2}^{0}\frac{dx}{x^3}+\int_{0}^{2}\frac{dx}{x^3}.$$

Using the definition of an improper integral, we obtain:

$$\int_{-2}^{2}\frac{dx}{x^3}=\int_{-2}^{0}\frac{dx}{x^3}+\int_{0}^{2}\frac{dx}{x^3}=\lim_{\varepsilon\to0+}\int_{-2}^{-\varepsilon}\frac{dx}{x^3}+\lim_{\varepsilon\to0+}\int_{\varepsilon}^{2}\frac{dx}{x^3}.$$

For the first integral,

$$\lim_{\varepsilon\to0+}\int_{-2}^{-\varepsilon}\frac{dx}{x^3}=\lim_{\varepsilon\to0+}\left(\frac{x^{-2}}{-2}\right)\Bigg|_{-2}^{-\varepsilon}=-\frac{1}{2}\lim_{\varepsilon\to0+}\frac{1}{x^2}\Bigg|_{-2}^{-\varepsilon}$$

$$=-\frac{1}{2}\lim_{\varepsilon\to0+}\left(\frac{1}{(-\varepsilon)^2}-\frac{1}{(-2)^2}\right)=-\frac{1}{2}\lim_{\varepsilon\to0+}\frac{1}{\varepsilon^2}+\frac{1}{8}=\infty.$$

Hence, the initial integral diverges.

Example 193.

Determine whether the integral $\int_{1}^{\infty}\frac{x+1}{\sqrt{x}}dx$ converges or diverges.

Solution.

Note, that

$$\frac{x+1}{\sqrt{x}}>\frac{x}{\sqrt{x}}=\sqrt{x}.$$

Since the improper integral $\int_{1}^{\infty}\sqrt{x}dx$ diverges, the given integral

$\int_{1}^{\infty}\frac{x+1}{\sqrt{x}}dx$ also diverges by Comparison Theorem 2.

Example 194.

Determine whether the integral $\int_0^{\infty} e^{-x}\sin x dx$ converges or diverges.

Solution.

Note, that $\left|e^{-x}\sin x\right| \le \left|e^{-x}\right|$

Show, that the integral $\int_1^{\infty}\left|e^{-x}\right|dx$ is convergent.

$$\int_1^{\infty}\left|e^{-x}\right|dx = \int_1^{\infty} e^{-x}dx = \lim_{n\to\infty}\int_1^{n} e^{-x}dx = \lim_{n\to\infty}\left(-e^{-x}\right)\Big|_1^n$$

$$= -\lim_{n\to\infty}\left(e^{-n} - e^{-1}\right) = -\left(0 - \frac{1}{e}\right) = \frac{1}{e}.$$

Then, by Comparison Theorems 1 and 3, the given integral $\int_0^{\infty} e^{-x}\sin x dx$ is also convergent.

Example 195.

Determine whether the integral $\int_1^{2}\frac{dx}{\sqrt{x-1}}$ converges or diverges.

Solution.

There is a discontinuity at $x = 1$ so that we must write the improper integral as

$$\int_1^{2}\frac{dx}{\sqrt{x-1}} = \lim_{\varepsilon\to 0+}\int_{1+\varepsilon}^{2}\frac{dx}{\sqrt{x-1}} = \lim_{\varepsilon\to 0+}\left(2\sqrt{x-1}\right)\Big|_{1+\varepsilon}^{2}$$

$$= 2\lim_{\varepsilon\to 0+}\left(\sqrt{2-1} - \sqrt{1+\varepsilon-1}\right) = 2\lim_{\varepsilon\to 0+}\left(1-\sqrt{\varepsilon}\right) = 2.$$

Hence, the integral converges.

Example 196.

Determine whether the integral $\int_0^4 \frac{dx}{(x-2)^3}$ converges or diverges.

Solution.

There is a discontinuity at $x = 2$ so that we must consider two improper integrals:

$$\int_0^4 \frac{dx}{(x-2)^3} = \int_0^2 \frac{dx}{(x-2)^3} + \int_2^4 \frac{dx}{(x-2)^3}.$$

Using the definition of an improper integral, we obtain

$$\int_0^2 \frac{dx}{(x-2)^3} + \int_2^4 \frac{dx}{(x-2)^3} = \lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} \frac{dx}{(x-2)^3} + \lim_{\varepsilon\to 0+} \int_{2+\varepsilon}^4 \frac{dx}{(x-2)^3}.$$

For the first integral,

$$\lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} \frac{dx}{(x-2)^3} = \lim_{\varepsilon\to 0+} \int_0^{2-\varepsilon} (x-2)^{-3}\, d(x-2)$$

$$= \lim_{\varepsilon\to 0+} \left[\frac{(x-2)^{-3+1}}{-3+1} \right]\Bigg|_0^{2-\varepsilon} = -\frac{1}{2} \lim_{\varepsilon\to 0+} \left[\frac{1}{(x-2)^2} \right]\Bigg|_0^{2-\varepsilon}$$

$$= -\frac{1}{2} \lim_{\varepsilon\to 0+} \left[\frac{1}{(2-\varepsilon-2)^2} - \frac{1}{(0-2)^2} \right] = -\frac{1}{2} \lim_{\varepsilon\to 0+} \left[\frac{1}{\varepsilon^2} - \frac{1}{4} \right] = \infty.$$

Since it is divergent, the given integral $\int_0^4 \frac{dx}{(x-2)^3}$ is also divergent.

Example 197.

Determine for what values of k the integral $\int_0^1 \frac{dx}{x^k}$ $(k > 0,\ k \neq 1)$ converges.

Solution.

The integrand has discontinuity at x = 0 so that we can write the integral as

$$\int_0^1 \frac{dx}{x^k} = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 \frac{dx}{x^k} = \lim_{\varepsilon\to 0+}\int_\varepsilon^1 x^{-k}dx = \lim_{\varepsilon\to 0+}\left(\frac{x^{-k+1}}{-k+1}\right)\Bigg|_\varepsilon^1 = \frac{1}{1-k}\cdot\lim_{\varepsilon\to 0+}\left(x^{-k+1}\right)\Big|_\varepsilon^1$$

$$= \frac{1}{1-k}\cdot\lim_{\varepsilon\to 0+}\left(1^{-k+1} - \varepsilon^{-k+1}\right) = \frac{1}{1-k}\cdot\lim_{n\to\infty}\left(1-n^{1-k}\right).$$

As seen from the expression, there are 2 cases:

- If $0<k<1$, then $\lim\limits_{\varepsilon\to 0+} n^{1-k} = 0$ and the integral converges;
- If $k>1$, then $\lim\limits_{\varepsilon\to 0+} n^{1-k} = \lim\limits_{\varepsilon\to 0+}\frac{1}{n^{k-1}} = \infty$ and the integral diverges.

Example 198.

Determine whether the integral $\int_2^4 \frac{dx}{(x-3)^{7/5}}$ converges or diverges.

Solution.

There is a discontinuity at x = 3 so that we must consider two improper integrals:

$$\int_2^4 \frac{dx}{(x-3)^{7/5}} = \int_2^3 \frac{dx}{(x-3)^{7/5}} + \int_3^4 \frac{dx}{(x-3)^{7/5}}.$$

Using the definition of an improper integral, we obtain

$$\int_2^3 \frac{dx}{(x-3)^{7/5}} + \int_3^4 \frac{dx}{(x-3)^{7/5}} = \lim_{\varepsilon\to 0+}\int_2^{3-\varepsilon} \frac{dx}{(x-3)^{7/5}} + \lim_{\varepsilon\to 0+}\int_{3-\varepsilon}^4 \frac{dx}{(x-3)^{7/5}}.$$

For the first integral,

$$\lim_{\varepsilon\to0+}\int_2^{3-\varepsilon}\frac{dx}{(x-3)^{7/5}}=\lim_{\varepsilon\to0+}\int_2^{3-\varepsilon}(x-3)^{-\frac{7}{5}}dx=\lim_{\varepsilon\to0+}\left[\frac{(x-3)^{-\frac{7}{5}+1}}{-\frac{7}{5}+1}\right]\Bigg|_2^{3-\varepsilon}$$

$$=-\frac{5}{2}\lim_{\varepsilon\to0+}\left[(x-3)^{-\frac{2}{5}}\right]\Bigg|_2^{3-\varepsilon}=-\frac{5}{2}\lim_{\varepsilon\to0+}\left[\frac{1}{(x-3)^{\frac{2}{5}}}\right]\Bigg|_2^{3-\varepsilon}$$

$$=-\frac{5}{2}\lim_{\varepsilon\to0+}\left[\frac{1}{(3-\varepsilon-3)^{\frac{2}{5}}}-\frac{1}{(2-3)^{\frac{2}{5}}}\right]$$

$$=-\frac{5}{2}\lim_{\varepsilon\to0+}\left[\frac{1}{(-\varepsilon)^{\frac{2}{5}}}-\frac{1}{(-1)^{\frac{2}{5}}}\right]=\infty.$$

Thus, the given integral $\int_2^4\frac{dx}{(x-3)^{7/5}}$ diverges.

Example 199.

Find the area under the curve $y=\ln x$ between $x=0$ and $x=1$.

Solution.

The given region is sketched in Figure 13. Since it is infinite, calculate the improper integral $\int_0^1 \ln x dx$ to find the area: $\lim_{\varepsilon\to0+}\int_\varepsilon^1 \ln x dx$.

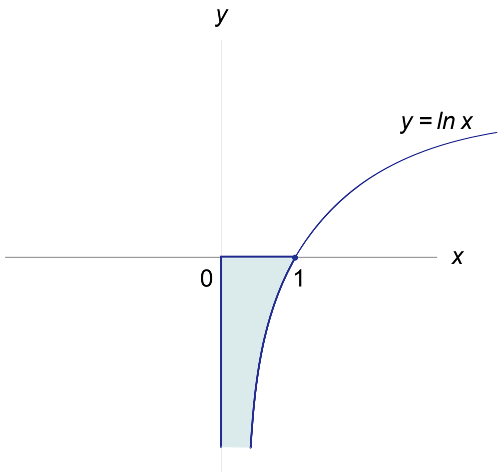

Figure 13

Use integration by parts. Let $u = \ln x$, $dv = dx$. Then $du = \dfrac{dx}{x}$, $v = x$. Thus

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \ln x dx = \lim_{\varepsilon \to 0+} \left[x \ln x \Big|_{\varepsilon}^{1} - \int_{\varepsilon}^{1} x \frac{dx}{x} \right] = \lim_{\varepsilon \to 0+} \left[x \ln x - x \right]_{\varepsilon}^{1}$$

$$= \lim_{\varepsilon \to 0+} \left[(\ln 1 - 1) - (\varepsilon \ln \varepsilon - \varepsilon) \right] = (0 - 1) - \lim_{\varepsilon \to 0+} \varepsilon (\ln \varepsilon - 1)$$

$$= -1 - \lim_{\varepsilon \to 0+} \frac{\ln \varepsilon - 1}{1/\varepsilon}.$$

We can apply L'Hopital's rule to find that

$$\lim_{\varepsilon \to 0+} \frac{\ln \varepsilon - 1}{1/\varepsilon} = \lim_{\varepsilon \to 0+} \frac{1/\varepsilon}{-1/\varepsilon^2} = -\lim_{\varepsilon \to 0+} \frac{\varepsilon^2}{\varepsilon} = -\lim_{\varepsilon \to 0+} \varepsilon = 0.$$

Hence, the improper integral is

$$\lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \ln x dx = -1 - \lim_{\varepsilon \to 0+} \frac{\ln \varepsilon - 1}{1/\varepsilon} = -1 - 0 = -1.$$

As seen from the figure, the required area is $S = \left| \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{1} \ln x dx \right| = 1$.

Example 200.

Find the area under the curve $y = \tan x$ between $x = 0$ and $x = \frac{\pi}{2}$.

Solution.

The required region is sketched in Figure 14. Since it is infinite, calculate the improper integral $\int_0^{\pi/2} \tan x dx$:

$$S = \int_0^{\pi/2} \tan x dx = \lim_{\varepsilon \to 0+} \int_0^{\frac{\pi}{2}-\varepsilon} \tan x dx = \lim_{\varepsilon \to 0+} \left(-\ln \cos x \right) \Big|_0^{\frac{\pi}{2}-\varepsilon}$$

$$= -\lim_{\varepsilon \to 0+} \left(\ln \cos\left(\frac{\pi}{2} - \varepsilon \right) - \ln \cos 0 \right) = \infty.$$

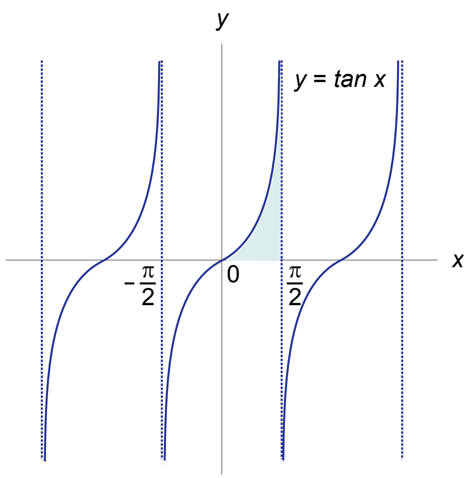

Figure 14

Thus, the improper integral is divergent and, hence, the area is equal to infinity.