Calculus

Techniques of Differentiation

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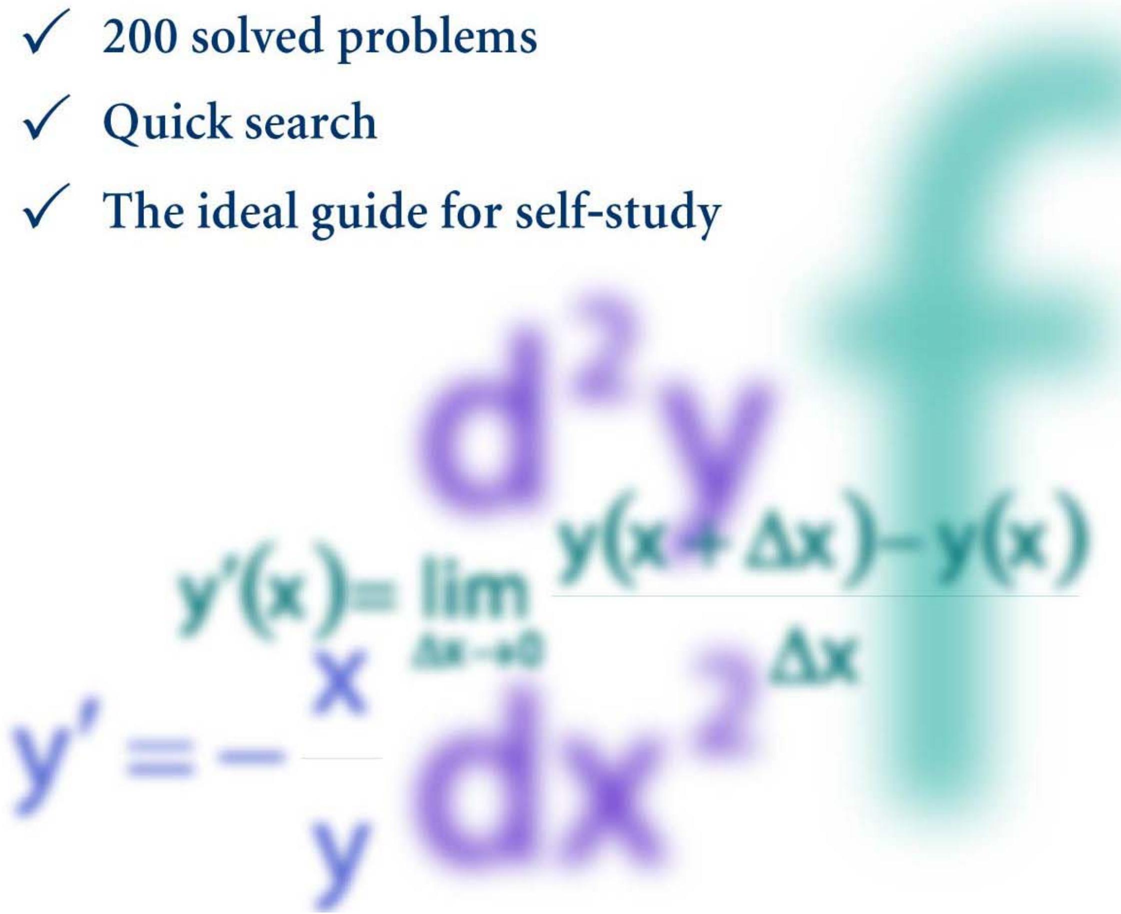
- ✓ Differentiation formulas
- ✓ 200 solved problems
- ✓ Quick search
- ✓ The ideal guide for self-study



Calculus
Techniques of Differentiation

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Preface

This ebook is intended primarily for students learning calculus and focuses entirely on differentiation of functions of one variable. The sections are written as self-guided tutorials. Each chap ter begins with appropriate definitions and formulas followed by numerous solved problems listed in order of increasing difficulty. Basic high school math is all that's needed to follow the explanations and learn from 200 practical problems solved step-by-step.

It is well known that the only way to learn calculus is by solving problems. The more problems you work, the better you become at solving them.This ebook helps you cut study time, increase problem-solving skills and achieve your personal best on calculus exams!

Contents

Chapter 4

The Product and Quotient Rules

The Product Rule

Let $u(x)$ and $v(x)$ be differentiable. Then uv is also differentiable and

$$(uv)' = u'v + uv'.$$

Important! The derivative of the product is NOT equal to the product of the derivatives.

The Quotient Rule

Let $u(x)$ and $v(x)$ be differentiable. Then, if $v(x) \neq 0$, we have

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}.$$

SOLVED PROBLEMS

Find the derivatives of the following functions (examples 46-70):

Example 46.

$y(x) = (x^4 + 2x)(x^3 + x).$

Solution.

Use the product rule and the power rule:

$$y'(x)=\left[\left(x^4+2x\right)\left(x^3+x\right)\right]'=\left(x^4+2x\right)'\left(x^3+x\right)+\left(x^4+2x\right)\left(x^3+x\right)'$$
$$=4x^3\cdot\left(x^3+x\right)+\left(x^4+2x\right)\cdot 3x^2=4x^6+4x^4+3x^6+6x$$
$$=x\left(7x^5+4x^3+6\right).$$

Example 47.

Differentiate $x^{5/2}$ by writing

$$x^{\frac{5}{2}}=x^{\frac{1}{2}}x^2=\sqrt{x}\cdot x^2$$

and using the product rule.

Solution.

$$\left(x^{\frac{5}{2}}\right)'=\left(\sqrt{x}\cdot x^2\right)'=\left(\sqrt{x}\right)'\cdot x^2+\left(\sqrt{x}\right)\cdot\left(x^2\right)'$$
$$=\frac{1}{2\sqrt{x}}\cdot x^2+\left(\sqrt{x}\right)\cdot 2x=\frac{x^2}{2\sqrt{x}}+2x\sqrt{x}$$
$$=\frac{1}{2}x^{2-\frac{1}{2}}+2x^{1+\frac{1}{2}}=\frac{1}{2}x^{\frac{3}{2}}+2x^{\frac{3}{2}}=\frac{5}{2}x^{\frac{3}{2}}.$$

Example 48.

$$y(x)=\frac{2}{x}.$$

Solution.

Compute the derivative of this function using the quotient rule:

$$y'(x)=\left(\frac{2}{x}\right)'=\frac{21\cdot x-2\cdot x'}{x^2}=\frac{0\cdot x-2\cdot 1}{x^2}=\frac{0-2}{x^2}=-\frac{2}{x^2}.$$

Example 49.

Find a formula for the derivative of a negative power
$y(x) = x^{-n}$.

Solution.

We can write the function as

$$y(x) = \frac{1}{x^n}.$$

Hence, by the quotient rule, the derivative is

$$y'(x) = \left(\frac{1}{x^n}\right)' = \frac{1' \cdot x^n - 1 \cdot (x^n)'}{(x^n)^2}$$

$$= \frac{0 \cdot x^n - nx^{n-1}}{x^{2n}} = -\frac{n}{x^{2n-n+1}} = -\frac{n}{x^{n+1}}.$$

Example 50.

$$f(x) = \frac{2x-3}{3x+2}.$$

Solution.

By the quotient rule,

$$f'(x) = \left(\frac{2x-3}{3x+2}\right)' = \frac{(2x-3)'(3x+2) - (2x-3)(3x+2)'}{(3x+2)^2}$$

$$= \frac{2 \cdot (3x+2) - (2x-3) \cdot 3}{(3x+2)^2} = \frac{6x+4-6x+9}{(3x+2)^2} = \frac{13}{(3x+2)^2}.$$

Example 51.

$y(x) = (x+a)(x-b)$, where a, b are constants.

Solution.

Use the product rule:

$$y'(x)=[(x+a)(x-b)]'=(x+a)'(x-b)+(x+a)(x-b)'$$
$$=1\cdot(x-b)+(x+a)\cdot 1=x-b+x+a=2x-b+a.$$

Example 52.

$$y(x)=\frac{2x}{1-x^2}.$$

Solution.

Use the quotient rule:

$$y'(x)=\left(\frac{2x}{1-x^2}\right)'=\frac{(2x)'(1-x^2)-(2x)(1-x^2)'}{(1-x^2)^2}.$$

Differentiate the expressions in the brackets and simplify:

$$y'(x)=\frac{2(1-x^2)-(2x)(-2x)}{(1-x^2)^2}=\frac{2-2x^2+4x^2}{(1-x^2)^2}=\frac{2(1+x^2)}{(1-x^2)^2}.$$

Example 53.

$$y(x)=\frac{1+x-x^2}{1-x+x^2}.$$

Solution.

Use the quotient rule:

$$y'(x)=\left(\frac{1+x-x^2}{1-x+x^2}\right)'$$
$$=\frac{(1+x-x^2)'(1-x+x^2)-(1+x-x^2)(1-x+x^2)'}{(1-x+x^2)^2}.$$

Differentiate the functions in the nominator:

$$y'(x)=\frac{(1-2x)(1-x+x^2)-(1+x-x^2)(-1+2x)}{(1-x+x^2)^2}.$$

Simplify:

$$y'(x)=\frac{(1-x+x^2-2x+2x^2-2x^3)-(-1-x+x^2+2x+2x^2-2x^3)}{(1-x+x^2)^2}$$

$$=\frac{1-x+x^2-2x+2x^2-2x^3+1+x-x^2-2x-2x^2+2x^3}{(1-x+x^2)^2}$$

$$=\frac{2-4x}{(1-x+x^2)^2}.$$

Example 54.

Find the derivative of tangent function $y(x)=\tan x$ by using the quotient rule.

Solution.

$\tan x=\frac{\sin x}{\cos x}$, so that, using the quotient rule, we obtain

$$y'(x)=(\tan x)'=\left(\frac{\sin x}{\cos x}\right)'=\frac{(\sin x)'\cos x-\sin x(\cos x)'}{(\cos x)^2}.$$

Since $(\sin x)'=\cos x,\quad (\cos x)'=-\sin x$, we have

$$(\tan x)'=\frac{\cos x\cos x-\sin x(-\sin x)}{(\cos x)^2}=\frac{\cos^2 x+\sin^2 x}{(\cos x)^2}=\frac{1}{\cos^2 x}.$$

Here we used the trigonometric identity

$\sin^2 x+\cos^2 x=1$.

Example 55.

Find the derivative of cotangent function $y(x)=\cot x$ using the quotient rule.

Solution.

We can write

$$y(x) = \cot x = \frac{\cos x}{\sin x}.$$

By quotient rule, the derivative is

$$(\cot x)' = \left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \sin x - \cos x (\sin x)'}{(\sin x)^2}.$$

Since $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, we have

$$(\cot x)' = \frac{-\sin x \sin x - \cos x \cos x}{(\sin x)^2} = -\frac{\cos^2 x + \sin^2 x}{(\sin x)^2} = -\frac{1}{\sin^2 x}.$$

Here we used the trigonometric identity

$$\sin^2 x + \cos^2 x = 1.$$

Example 56.

$$y(x) = \frac{\sin x}{1 + \cos x}.$$

Solution.

Use the quotient rule:

$$y'(x) = \left(\frac{\sin x}{1 + \cos x}\right)' = \frac{(\sin x)'(1 + \cos x) - \sin x (1 + \cos x)'}{(1 + \cos x)^2}.$$

Since $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, we have

$$y'(x) = \frac{\cos x (1 + \cos x) - \sin x(-\sin x)}{(1 + \cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1 + \cos x)^2} = \frac{1 + \cos x}{(1 + \cos x)^2} = \frac{1}{1 + \cos x}.$$

We used here the trigonometric identity

$$\sin^2 x + \cos^2 x = 1$$

to simplify the answer.

Example 57.

Let $y(x)=\sin^2 x$. Differentiate the given function not using the chain rule.

Solution.

We can write

$y(x)=\sin x \sin x$.

By the product rule,

$y'(x)=(\sin x \sin x)' = (\sin x)' \sin x + \sin x (\sin x)'$.

Since $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, we have

$y'(x)=\cos x \sin x + \sin x \cos x = 2\sin x \cos x$.

Use the double angle formula

$\sin 2x = 2\sin x \cos x$.

So that the derivative is

$y'(x)=\sin 2x$.

Example 58.

$f(x)=\dfrac{ax+b}{cx+d}$, where a, b, c, and d are constants.

Solution.

Use the quotient rule:

$$f'(x)=\left(\frac{ax+b}{cx+d}\right)' = \frac{(ax+b)'(cx+d)-(ax+b)(cx+d)'}{(cx+d)^2}$$
$$= \frac{a\cdot(cx+d)-(ax+b)\cdot c}{(cx+d)^2} = \frac{acx+ad-acx-bc}{(cx+d)^2} = \frac{ad-bc}{(cx+d)^2}.$$

Write the nominator using determinant:

$$ad - bc = \begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

So that the derivative is

$$f'(x) = \left(\frac{ax+b}{cx+d}\right)' = \frac{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}{(cx+d)^2}.$$

Example 59.

$y(x) = (x-a)(x^2+ax+a^2)$, where a is a constant.

Solution.

Apply the product rule and simplify:

$$\begin{aligned} y'(x) &= \left[(x-a)(x^2+ax+a^2)\right]' \\ &= (x-a)'(x^2+ax+a^2) + (x-a)(x^2+ax+a^2)' \\ &= 1\cdot(x^2+ax+a^2) + (x-a)\cdot(2x+a) \\ &= x^2+ax+a^2+2x^2-2ax+ax-a^2 = 3x^2. \end{aligned}$$

Example 60.

$s(t) = (t^2+2t+8)(t^4-5t^3)$.

Solution.

Apply the product rule to compute the derivative.

$$\begin{aligned} s'(t) &= \left[(t^2+2t+8)(t^4-5t^3)\right]' \\ &= (t^2+2t+8)'(t^4-5t^3) + (t^2+2t+8)(t^4-5t^3)' \\ &= (2t+2)(t^4-5t^3) + (t^2+2t+8)(4t^3-15t^2) \\ &= (2t^5+2t^4-10t^4-10t^3) \\ &+ (4t^5+8t^4+32t^3-15t^4-30t^3-120t^2) \end{aligned}$$

$$= \left(2t^5 - 8t^4 - 10t^3\right) + \left(4t^5 - 7t^4 + 2t^3 - 120t^2\right)$$
$$= 6t^5 - 15t^4 - 8t^3 - 120t^2 = t^2\left(6t^3 - 15t^2 - 8t - 120\right).$$

Example 61.

$y(x) = 2\sin x\cos x$.

Solution.

Apply the product rule:

$$y'(x) = 2\left[(\sin x)'\cos x + \sin(\cos x)'\right].$$

Since $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, we obtain

$$y'(x) = 2(\cos x\cos x + \sin x(-\sin x)) = 2\left(\cos^2 x - \sin^2 x\right).$$

Simplify the answer using the double angle formula:

$\cos 2x = \cos^2 x - \sin^2 x$.

Then

$y'(x) = 2\cos 2x$.

(The answer is obvious because the original function can be writ ten as $y(x) = 2\sin x\cos x = \sin 2x$).

Example 62.

$y(x) = (x\sin\alpha + \cos\alpha)(x\cos\alpha - \sin\alpha)$, where α is a constant angle.

Solution.

Here $\cos\alpha$ and $\sin\alpha$ are constants. Then by the product rule,

$$y'(x) = [(x\sin\alpha + \cos\alpha)(x\cos\alpha - \sin\alpha)]'$$
$$= (x\sin\alpha + \cos\alpha)'(x\cos\alpha - \sin\alpha)$$
$$+ (x\sin\alpha + \cos\alpha)(x\cos\alpha - \sin\alpha)'$$
$$= (\sin\alpha + 0)(x\cos\alpha - \sin\alpha) + (x\sin\alpha + \cos\alpha)(\cos\alpha - 0)$$

$$= \sin\alpha(x\cos\alpha - \sin\alpha) + (x\sin\alpha + \cos\alpha)\cos\alpha$$
$$= x\sin\alpha\cos\alpha - \sin^2\alpha + x\sin\alpha\cos\alpha + \cos^2\alpha$$
$$= 2\sin\alpha\cos\alpha \cdot x + \cos^2\alpha - \sin^2\alpha\,.$$

Use the following trigonometric identities:

$\sin 2\alpha = 2\sin\alpha\cos\alpha$,

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.

Then we have

$y'(x) = \sin 2\alpha \cdot x + \cos 2\alpha$.

Example 63.

$y(x) = x(2x+1)(3x+2)$.

Solution.

We can find the derivative of the given function by multiplying out first. In this case, we have

$$y(x) = x(2x+1)(3x+2) = x(6x^2 + 3x + 4x + 2)$$
$$= 6x^3 + 7x^2 + 2x.$$

The derivative of this is

$$y'(x) = (6x^3 + 7x^2 + 2x)' = 18x^2 + 14x + 2\,.$$

Now apply the product rule (twice) to differentiate this function.

$$y'(x) = [x(2x+1)(3x+2)]'$$
$$= [x(2x+1)]' \cdot (3x+2) + [x(2x+1)] \cdot (3x+2)'$$
$$= \left[x' \cdot (2x+1) + x \cdot (2x+1)'\right] \cdot (3x+2) + [x(2x+1)] \cdot 3$$
$$= [1 \cdot (2x+1) + x \cdot 2](3x+2) + 3x(2x+1)$$
$$= [2x + 1 + 2x](3x+2) + 6x^2 + 3x$$
$$= (4x+1)(3x+2) + 6x^2 + 3x$$
$$= 12x^2 + 3x + 8x + 2 + 6x^2 + 3x$$
$$= 18x^2 + 11x + 2\,.$$

It is easy to show (see Problem **64**), that if a function is a factor of 3 other functions:

$f(x) = u(x)v(x)w(x)$,

its derivative is

$f'(x) = u'vw + uv'w + uvw'$.

This formula allows to find the given derivative directly.

Example 64.

Let $f(x) = u(x)v(x)w(x)$. Find a formula for f '(x).

Solution.

Apply the product rule twice. Then the derivative is

$$f'(x) = (u(x)v(x)w(x))' = [u(x)v(x)]'w(x) + [u(x)v(x)]w(x)'$$

Since

$[u(x)v(x)]' = u'v + uv'$,

we obtain the following formula

$$f'(x) = (uvw)' = (u'v + uv')w + uvw' = u'vw + uv'w + uvw'.$$

Example 65.

Differentiate the following function not using the chain rule.

$f(x) = (x^2 - 1)^3$.

Solution.

Use the formula given in Problem **64**:

$(uvw)' = u'vw + uv'w + uvw'$.

We can write the function as

$f(x) = (x^2 - 1)(x^2 - 1)(x^2 - 1)$.

The derivative is

$$f'(x) = (x^2 - 1)'(x^2 - 1)(x^2 - 1)$$

$$+\left(x^2-1\right)\left(x^2-1\right)'\left(x^2-1\right)+\left(x^2-1\right)\left(x^2-1\right)\left(x^2-1\right)'$$
$$=2x\left(x^2-1\right)\left(x^2-1\right)+\left(x^2-1\right)\cdot 2x\cdot\left(x^2-1\right)+\left(x^2-1\right)\left(x^2-1\right)\cdot 2x$$
$$=3\cdot 2x\left(x^2-1\right)\left(x^2-1\right)=6x\left(x^2-1\right)^2.$$

Example 66.

$y(x)=\left(x^2+3x+4\right)\left(2x-5\right)\left(x^2+3\right).$

Solution.

Use the formula given in Problem **64**:

$(uvw)'=u'vw+uv'w+uvw'.$

Supposing that

$u(x)=x^2+3x+4,$

$v(x)=2x-5,$

$w(x)=x^2+3,$

we have

$$y'(x)=\left(x^2+3x+4\right)'(2x-5)\left(x^2+3\right)+\left(x^2+3x+4\right)(2x-5)'\left(x^2+3\right)$$
$$+\left(x^2+3x+4\right)(2x-5)\left(x^2+3\right)'$$
$$=(2x+3)(2x-5)\left(x^2+3\right)+\left(x^2+3x+4\right)\cdot 2\cdot\left(x^2+3\right)$$
$$+\left(x^2+3x+4\right)(2x-5)\cdot 2x.$$

Simplify the expression:

$$y'(x)=\left(4x^2-4x-15\right)\left(x^2+3\right)+\left(x^2+3x+4\right)\left(2x^2+6\right)$$
$$+\left(x^2+3x+4\right)\left(4x^2-10x\right)$$
$$=\left(4x^4-4x^3-15x^2+12x^2-12x-45\right)$$
$$+\left(2x^4+6x^3+8x^2+6x^2+18x+24\right)$$
$$+\left(4x^4+12x^3+16x^2-10x^3-30x^2-40x\right).$$

So that

$y'(x)=10x^4+4x^3-3x^2-34x-21.$

Example 67.

Find a general formula to differentiate a reciprocal 1/f(x) of a function f(x).

Solution.

Let $y(x)=\frac{1}{f(x)}$. By the quotient rule, the derivative is

$$y'(x)=\left(\frac{1}{f(x)}\right)'=\frac{1'\cdot f(x)-1\cdot f'(x)}{f^2(x)}$$

$$=\frac{0\cdot f(x)-1\cdot f'(x)}{f^2(x)}=-\frac{f'(x)}{f^2(x)}.$$

So we get the reciprocal rule for derivative:

$$\left(\frac{1}{f(x)}\right)'=-\frac{f'(x)}{f^2(x)}.$$

Example 68.

$$y(x)=\frac{1}{x^6+5x^2}.$$

Solution.

Use the reciprocal rule in Example 67 to find the derivative:

$$\left(\frac{1}{f(x)}\right)'=-\frac{f'(x)}{f^2(x)}.$$

Hence

$$y'(x)=\left(\frac{1}{x^6+5x^2}\right)'=-\frac{\left(x^6+5x^2\right)'}{\left(x^6+5x^2\right)^2}=-\frac{6x^5+10x}{\left(x^6+5x^2\right)^2}.$$

Example 69.

$y(x) = \sec x$.

Solution.

Use the reciprocal rule for the derivative:

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)}.$$

The derivative is

$$y'(x) = (\sec x)' = \left(\frac{1}{\cos x}\right)' = -\frac{(\cos x)'}{\cos^2 x}$$

$$= -\frac{(-\sin x)}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \tan x \sec x.$$

Example 70.

$y(x) = \csc x$.

Solution.

Use the formula for the derivative of reciprocal function:

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{f^2(x)}.$$

The derivative is

$$y'(x) = (\csc x)' = \left(\frac{1}{\sin x}\right)' = -\frac{(\sin x)'}{\sin^2 x}$$

$$= -\frac{\cos x}{\sin^2 x} = -\frac{\cos x}{\sin^2 x} = -\cot x \csc x.$$