Calculus

Infinite Sequences and Series

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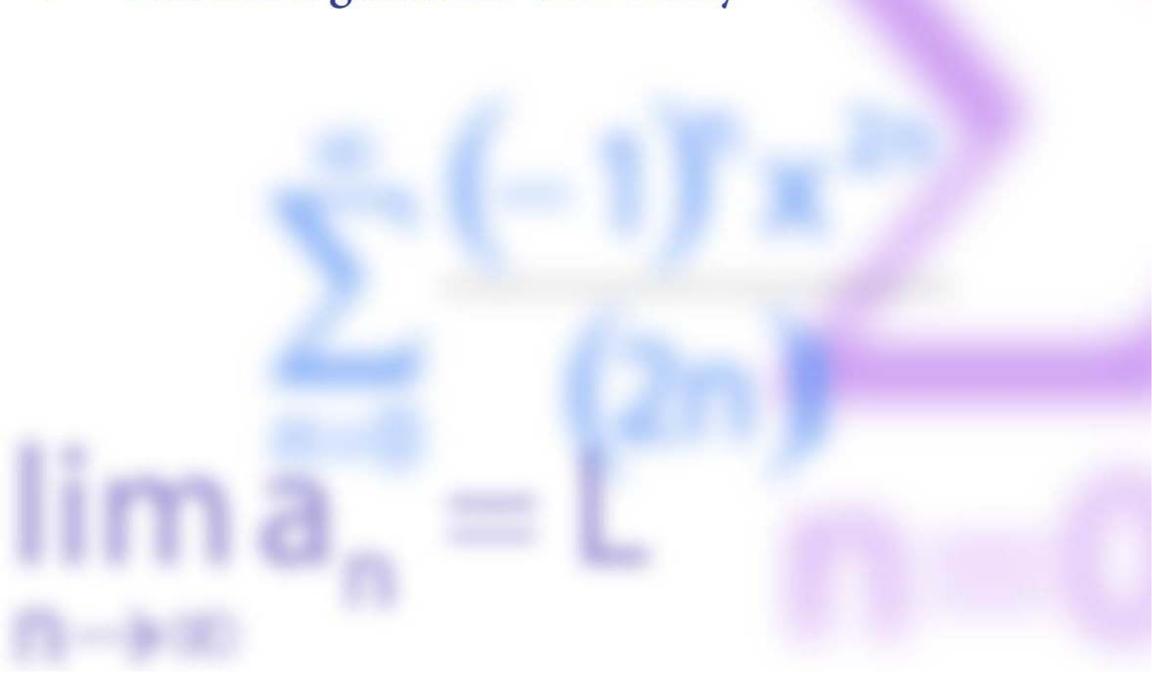
- ✓ 150 solved problems
- ✓ Formulas and convergence tests
- ✓ Quick search
- ✓ The ideal guide for self-study



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Preface

This ebook is intended for all students who study calculus, and considers 150 typical problems on infinite sequences and series, fully solved step-by-step.

Topics include: Infinite Sequences, Geometric Series, Infinite Series, Comparison Tests, Integral Test, Ratio and Root Tests, Absolute and Conditional Convergence of Alternating Series, Power Series, Differentiation and Integration of Power Series, Taylor and Maclaurin Series. Each of the 10 chapters includes appropriate definitions and formulas followed by solved problems listed in order of inreasing difficulty.

Studying and solving these problems helps you cut study time, increase problem-solving skills and achieve your personal best on calculus exams!

Contents

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Chapter 7

Alternating Series: Absolute and Conditional Convergence

A series in which successive terms have opposite signs is called an alternating series.

The Alternating Series Test (Leibniz's Theorem)

Let $\{a_n\}$ be a sequence of positive numbers such that

1. $a_{n+1} < a_n$ for all n.

2. $\lim_{n\to\infty} a_n = 0$

Then the alternating series $\sum_{n=1}^{\infty}(-1)^n a_n$ and $\sum_{n=1}^{\infty}(-1)^{n-1} a_n$ both converge.

Absolute and Conditional Convergence

A series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty}|a_n|$ is convergent.

If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent then it is convergent. The converse of this statement is false.

A series $\sum_{n=1}^{\infty} a_n$ is conditionally convergent if the series is convergent but is not absolutely convergent.

SOLVED PROBLEMS

Example 91.

Use the alternating series test to determine the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2 n}{n}$.

Solution.

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| (-1)^n \frac{\sin^2 n}{n} \right| = \lim_{n\to\infty} \frac{\sin^2 n}{n} = 0,$$

since $\sin^2 n \le 1$. Therefore, the given series converges.

Example 92.

Use the alternating series test to determine the convergence of the series $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+10}$.

Solution.

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| (-1)^n \frac{\sqrt{n}}{n+10} \right| = \lim_{n\to\infty} \frac{\sqrt{n}}{n+10}$$

$$= \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{n}}{\frac{n}{n} + \frac{10}{n}} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}}}{1 + \frac{10}{n}} = \frac{0}{1} = 0\,.$$

Hence, the given series converges.

Example 93.

Determine whether $\sum_{n=1}^{\infty} (-1)^n \frac{2n+1}{3n+2}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

We try to apply the alternating series test here.

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{2n+1}{3n+2} = \lim_{n\to\infty} \frac{\frac{2n+1}{n}}{\frac{3n+2}{n}} = \lim_{n\to\infty} \frac{2 + \frac{1}{n}}{3 + \frac{2}{n}} = \frac{2}{3} \neq 0\,.$$

Since the nth term does not approach 0, the given series diverges.

Example 94.

Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

Applying the ratio test to the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n!} \right|$ with nonnegative terms, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1} = 0.$$

Hence, the series is absolutely convergent.

Example 95.

Determine whether the given alternating series $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}\sqrt{n}}{\ln n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

First using the alternating series test, find the limit

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{\sqrt{n}}{\ln n}.$$

By L'Hopital's rule,

$$\lim_{n\to\infty}\frac{\sqrt{n}}{\ln n} = \lim_{n\to\infty}\frac{\frac{1}{2\sqrt{n}}}{\frac{1}{n}} = \frac{1}{2}\lim_{n\to\infty}\frac{n}{\sqrt{n}} = \frac{1}{2}\lim_{n\to\infty}\sqrt{n} = \infty.$$

Hence, the given series is divergent.

Example 96.

Use the alternating series test to determine the convergence of the series $\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{n^3+2}$.

Solution.

Since

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{(-1)^n n^2}{n^3+2}\right| = \lim_{n\to\infty}\frac{n^2}{n^3+2}$$

$$= \lim_{n\to\infty}\frac{\frac{n^2}{n^3}}{\frac{n^3+2}{n^3}} = \lim_{n\to\infty}\frac{\frac{1}{n}}{1+\frac{2}{n^3}} = 0,$$

the series is convergent.

Example 97.

Study the convergence of the series $\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{n!}$.

Solution.

Apply the ratio test to the series $\sum_{n=0}^{\infty}\left|(-1)^n\frac{3^n}{n!}\right| = \sum_{n=0}^{\infty}\frac{3^n}{n!}$ with non-negative terms.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\frac{3^{n+1}}{(n+1)!}}{\frac{3^n}{n!}} = \lim_{n\to\infty}\frac{3^{n+1}n!}{3^n(n+1)!}$$

$$= \lim_{n\to\infty}\frac{3\cdot 3^n n!}{3^n(n+1)n!} = \lim_{n\to\infty}\frac{3}{(n+1)} = 0.$$

Hence, the given series is absolutely convergent.

Example 98.

Determine the nth term of and test for convergence the series

$$\frac{2}{3!} - \frac{2^2}{5!} + \frac{2^3}{7!} - \frac{2^4}{9!} + \dots.$$

Solution.

The nth term of the series is $a_n = (-1)^{n+1} \dfrac{2^n}{(2n+1)!}$. Apply the ratio test to the series $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \dfrac{2^n}{(2n+1)!}$ with nonnegative terms.

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{\dfrac{2^{n+1}}{(2n+3)!}}{\dfrac{2^n}{(2n+1)!}} = \lim_{n\to\infty} \frac{2^{n+1}(2n+1)!}{2^n(2n+3)!}$$

$$= \lim_{n\to\infty} \frac{2^{n+1}(2n+1)!}{2^n(2n+3)!} = \lim_{n\to\infty} \frac{2}{(2n+2)(2n+3)} = 0.$$

Thus, the given series is absolutely convergent.

Example 99.

Determine whether the alternating series $\sum_{n=1}^{\infty} \dfrac{(-1)^n n^3}{n^3 + 5n^2 - 13n - 2}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

Applying the alternating series test, we have

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^n n^3}{n^3 + 5n^2 - 13n - 2} \right| = \lim_{n\to\infty} \frac{n^3}{n^3 + 5n^2 - 13n - 2}$$

$$= \lim_{n\to\infty} \frac{\dfrac{n^3}{n^3}}{\dfrac{n^3}{n^3} + \dfrac{5n^2}{n^3} - \dfrac{13n}{n^3} - \dfrac{2}{n^3}} = \lim_{n\to\infty} \frac{1}{1 + \dfrac{5}{n} - \dfrac{13}{n^2} - \dfrac{2}{n^3}} = 1.$$

Since the nth term does not approach 0, the original series is divergent.

Example 100.

Determine whether the alternating series $\sum_{n=1}^{\infty}\frac{(-3)^n}{n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

By the alternating series test,

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{(-3)^n}{n}\right| = \lim_{n\to\infty}\left|\frac{3^n}{n}\right|.$$

Use the L'Hopital's rule to calculate this limit.

$$\lim_{n\to\infty}\left|\frac{3^n}{n}\right| = \lim_{n\to\infty}\frac{3^n \ln 3}{1} = \infty .$$

Therefore, the given alternating series diverges.

Example 101.

Determine whether $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{5n-1}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

By the alternating series test,

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{5n-1}\right| = \lim_{n\to\infty}\frac{1}{5n-1} = 0,$$

and the series is convergent.

Now consider the convergence of the series $\sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{5n-1}\right| = \sum_{n=1}^{\infty}\frac{1}{5n-1}$ with nonnegative terms. Using the integral test, we obtain

$$\int_1^{\infty}\frac{dx}{5x-1} = \lim_{n\to\infty}\int_1^{n}\frac{dx}{5x-1} = \lim_{n\to\infty}\left[\frac{1}{5}\ln|5x-1|\Big|_1^n\right]$$

$$=\frac{1}{5}\lim_{n\to\infty}\left[\ln(5n-1)-\ln 4\right]=\infty .$$

Hence, the series $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{5n-1}$ is conditionally convergent.

Example 102.

Determine whether $\sum_{n=2}^{\infty}\frac{\sin n}{1-n^2}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

First we check that the series $\sum_{n=2}^{\infty}\frac{1}{1-n^2}$ is convergent. We compare $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ with the series $\sum_{n=2}^{\infty}\frac{1}{n^2}$. Using the limit comparison test, we have

$$\lim_{n\to\infty}\frac{\frac{1}{n^2-1}}{\frac{1}{n^2}}=\lim_{n\to\infty}\frac{n^2}{n^2-1}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{n^2}}=1 .$$

Since $\sum_{n=2}^{\infty}\frac{1}{n^2}$ is a p series with $p=2>1$, it is convergent. Then $\sum_{n=2}^{\infty}\frac{1}{n^2-1}$ is also convergent.

Note that $\left|\frac{\sin n}{1-n^2}\right|\le\frac{1}{n^2-1}$ for all $n\ge 2$.

By the comparison test, the series $\sum_{n=2}^{\infty}\frac{\sin n}{1-n^2}$ is absolutely convergent.

Example 103.

Study the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$ for convergence.

Solution.

First we apply the alternating series test:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\ln n} = 0.$$

Hence, this series converges. However, it is only conditionally convergent, since the series $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{\ln n}$ with nonnegative terms is divergent. We easily can see this by using the comparison test. Since $\ln n < n$ for $n \geq 2$, we obtain

$$\frac{1}{\ln n} > \frac{1}{n}.$$

But $\sum_{n=2}^{\infty} \frac{1}{n}$ is the harmonic series, which is divergent. Hence, the given series $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is also divergent by the comparison test.

Example 104.

Determine whether the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n(n+1)}}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

Apply the alternating series test:

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{\sqrt{n(n+1)}} = 0.$$

Hence, the given series is convergent. Now we determine is it absolutely or conditionally convergent. We will use the limit comparison test and compare the series $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ with the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}} = \lim_{n\to\infty}\frac{n}{\sqrt{n(n+1)}} = \lim_{n\to\infty}\frac{\sqrt{n^2}}{\sqrt{n(n+1)}}$$

$$= \lim_{n\to\infty}\sqrt{\frac{n^2}{n^2+n}} = \lim_{n\to\infty}\sqrt{\frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2}+\frac{n}{n^2}}} = \lim_{n\to\infty}\sqrt{\frac{1}{1+\frac{1}{n}}} = 1.$$

Since the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ is divergent, then the original series $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n(n+1)}}$ is conditionally convergent.

Example 105.

Determine whether the alternating series $\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n\ln n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution.

By the alternating series test,

$$\lim_{n\to\infty}|a_n| = \lim_{n\to\infty}\frac{1}{n\ln n} = 0,$$

so that it is convergent. Now we test it for absolute or conditional convergence. Applying the integral test to the series with non-negative terms, calculate the following improper integral

$$\int_2^{\infty}\frac{dx}{x\ln x}=\lim_{n\to\infty}\int_2^{n}\frac{dx}{x\ln x}=\lim_{n\to\infty}\int_2^{n}\frac{d\ln x}{\ln x}$$

$$=\lim_{n\to\infty}\ln\ln x\Big|_2^n=\lim_{n\to\infty}\left(\ln\ln n-\ln\ln 2\right)=\infty.$$

Hence, the original alternating series converges conditionally.