Calculus

Double Integrals

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- ✓ Integration formulas
- ✓ 50 solved problems
- ✓ Quick search
- ✓ The ideal guide for self-study

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Preface

This ebook is intended for all students who study multiple variable calculus, and considers 50 typical problems with use of double integrals, solved step-by-step. It is well-suited for use in independent study or as a reference.

Topics include: Definition and Properties of Double Integrals, It erated Integrals, Double Integrals over Rectangular Regions, Double Integrals over General Regions, Change of Variables in Double Integrals, Integration in Polar Coordinates, and Applications of Double Integrals. Each of the chapters includes appropriate definitions and formulas followed by solved problems.

Studying and solving these problems helps you cut study time, increase problems solving skills and achieve your personal best on calculus exams!

Contents

Chapter 6

Double Integrals in Polar Coordinates

We can also apply the change of variables to the polar coordinate transformation (see Figure 26):

$$x = r\cos\theta, \; y = r\sin\theta .$$

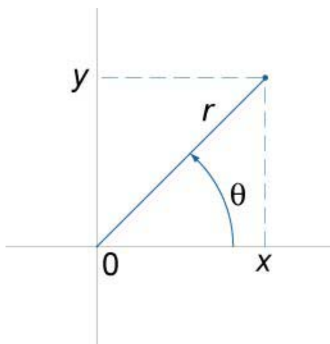

Figure 26.

The Jacobian determinant for this transformation is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \frac{\partial(r\cos\theta)}{\partial r} & \frac{\partial(r\cos\theta)}{\partial \theta} \\ \frac{\partial(r\sin\theta)}{\partial r} & \frac{\partial(r\sin\theta)}{\partial \theta} \end{vmatrix}$$

$$=\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}=\cos\theta\cdot r\cos\theta-(-r\sin\theta)\cdot\sin\theta$$

$$=r\cos^2\theta+r\sin^2\theta=r\left(\cos^2\theta+\sin^2\theta\right)=r\,.$$

As a result, the differential dxdy for polar coordinates is

$$dxdy=\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|drd\theta=rdrd\theta\,.$$

Let the region R is determined as follows (Figure 27):

$$0\le g(\theta)\le r\le h(\theta),\ \alpha\le\theta\le\beta\,,\text{ where }\ \beta-\alpha\le 2\pi\,.$$

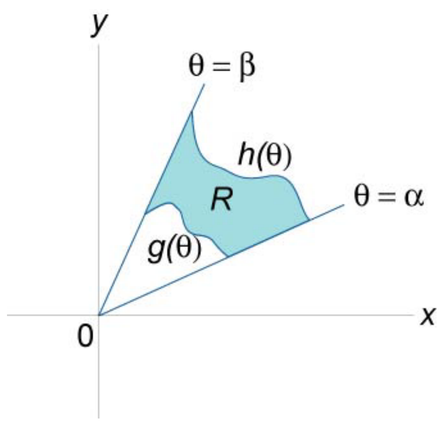

Figure 27.

Then we obtain the formula for change of variables in polar coordinates:

$$\iint_R f(x,y)\,dxdy = \int_{\alpha}^{\beta}\int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta\,.$$

If the region R is the so-called polar rectangle given by

$$0 \le a \le r \le b,\ \alpha \le \theta \le \beta,\ \text{where } \beta - \alpha \le 2\pi,$$

and shown in Figure 28.

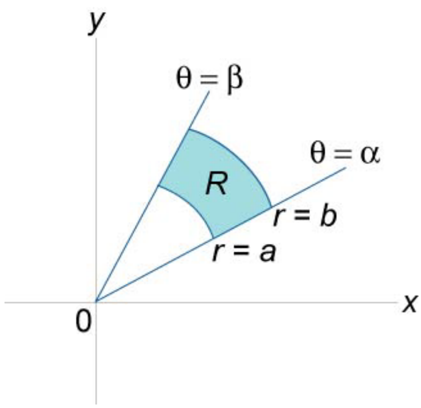

Figure 28.

Then the formula for change of variables can be written as

$$\iint_R f(x,y)\,dxdy = \int_{\alpha}^{\beta}\int_{a}^{b} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta\,.$$

Be careful not to forget the factor r (the Jacobian) in the right-hand side of the formula!

See also chapter 7 where we consider calculation of areas and volumes using polar coordinates.

SOLVED PROBLEMS

Example 31.

Calculate the integral by transforming to polar coordinates.

$\iint_R (x^2+y^2)dydx$, where R is the sector $0 \le \theta \le \frac{\pi}{2}$ of a circle with radius $r=\sqrt{3}$.

Solution.

The region of integration R is the polar rectangle

$R=\left\{(r,\theta) \mid 0 \le r \le \sqrt{3}, 0 \le \theta \le \frac{\pi}{2}\right\}$ (see Figure 29).

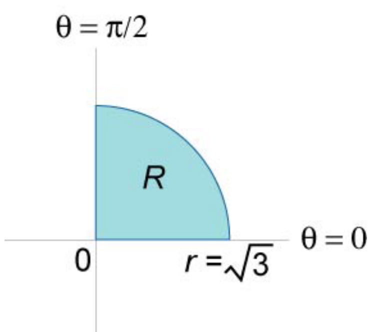

Figure 29.

So using the formula

$$\iint_R f(x,y)dxdy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)rdrd\theta,$$

we have

$$\iint_R (x^2+y^2)dydx = \int_0^{\pi/2} \int_0^{\sqrt{3}} r^2(\cos^2\theta + \sin^2\theta)rdrd\theta$$

$$= \int_0^{\pi/2} d\theta \int_0^{\sqrt{3}} r^3 dr = \theta\Big|_0^{\pi/2} \cdot \left(\frac{r^4}{4}\right)\Bigg|_0^{\sqrt{3}} = \frac{\pi}{2} \cdot \frac{9}{4} = \frac{9\pi}{8}.$$

Example 32.

Evaluate the integral $\iint_R xydydx$, where R is the region that lies between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 5$.

Solution.

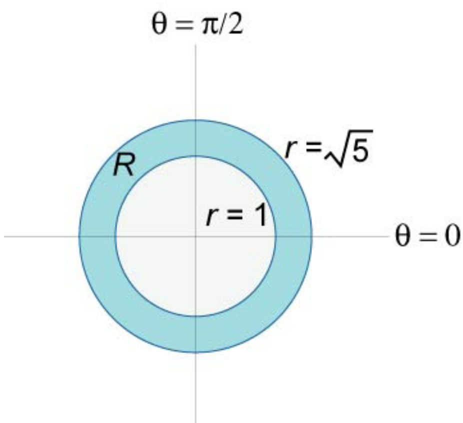

Figure 30.

In polar coordinates, the region of integration R is the polar rectangle (Figure 30):

$$R = \{(r,\theta) | 1 \le r \le \sqrt{5}, 0 \le \theta \le 2\pi\}.$$

So using the formula

$$\iint_R f(x,y)dxdy = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r dr d\theta,$$

we can write

$$\iint\limits_R xydydx = \int\limits_0^{2\pi}\int\limits_1^{\sqrt{5}} r\cos\theta r\sin\theta rdrd\theta$$

$$= \int\limits_0^{2\pi}\sin\theta\cos\theta d\theta\int\limits_1^{\sqrt{5}} r^3dr = \frac{1}{2}\int\limits_0^{2\pi}\sin 2\theta d\theta\int\limits_1^{\sqrt{5}} r^3dr$$

$$= \frac{1}{2}\left(-\frac{\cos 2\theta}{2}\right)\Bigg|_0^{2\pi}\cdot\left(\frac{r^4}{4}\right)\Bigg|_1^{\sqrt{5}} = \frac{1}{4}(-\cos 4\pi + \cos 0)\cdot\frac{1}{4}(25-1)$$

$$= \frac{1}{4}(-1+1)\cdot\frac{24}{4} = 6.$$

Example 33.

Evaluate $\iint\limits_R \sin\theta drd\theta$, where R is the region enclosed by the cardioid $r = 1+\cos\theta$.

Solution.

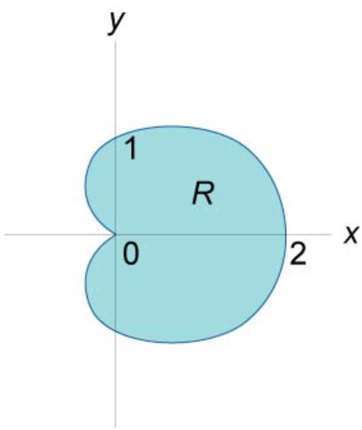

Figure 31.

In polar coordinates, the integral can be written as

$$\iint_R \sin\theta\, dr d\theta = \int_0^{2\pi} \int_0^{1+\cos\theta} dr d\theta = \int_0^{2\pi} \left[\int_0^{1+\cos\theta} dr \right] \sin\theta d\theta$$

$$= \int_0^{2\pi} \left[r\Big|_0^{1+\cos\theta} \right] \sin\theta d\theta = \int_0^{2\pi} (1+\cos\theta)\sin\theta d\theta$$

$$= \int_0^{2\pi} (\sin\theta + \cos\theta\sin\theta) d\theta = \int_0^{2\pi} \sin\theta d\theta + \int_0^{2\pi} \frac{\sin 2\theta}{2} d\theta$$

$$= (-\cos\theta)\Big|_0^{2\pi} + \frac{1}{2}\left(-\frac{\cos 2\theta}{2} \right)\Big|_0^{2\pi}$$

$$= -\cos 2\pi + \cos 0 - \frac{1}{4}\cos 4\pi + \frac{1}{4}\cos 0$$

$$= -1 + 1 - \frac{1}{4} + \frac{1}{4} = 0.$$

Example 34.

Calculate the double integral $\iint_R (x^2 + y^2) dxdy$, where R is the region enclosed by the circle $x^2 + y^2 = 2x$.

Solution.

The region of integration R is indicated in Figure 32.

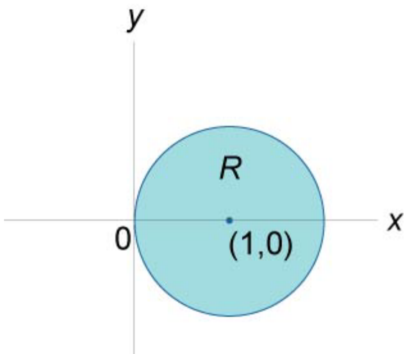

Figure 32.

We can transform the equation of the circle as

$x^2 + y^2 = 2x$,

$x^2 - 2x + 1 + y^2 = 1$,

$(x-1)^2 + y^2 = 1$.

Substituting the expressions $x = r\cos\theta$, $y = r\sin\theta$, we obtain the equation of the circle in polar coordinates.

$x^2 + y^2 = 2x$,

$r^2\cos^2\theta + r^2\sin^2\theta = 2r\cos\theta$,

$r^2(\cos^2\theta + \sin^2\theta) = 2r\cos\theta$,

$r = 2\cos\theta$.

The pullback S of the region R is shown in Figure 33.

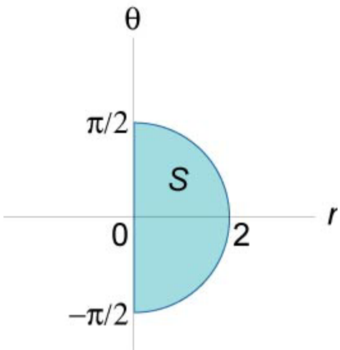

Figure 33.

Then the double integral in polar coordinates can be written as

$$\iint_R \left(x^2+y^2\right)dxdy = \iint_S \left(r^2\cos^2\theta + r^2\sin^2\theta\right) r\,dr\,d\theta$$

$$= \iint_S r^3\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\int_0^{2\cos\theta} r^3\,dr\right]d\theta = \int_{-\pi/2}^{\pi/2}\left[\left.\left(\frac{r^4}{4}\right)\right|_0^{2\cos\theta}\right]d\theta$$

$$= 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta = 4\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(1+2\cos 2\theta+\cos^2 2\theta\right)d\theta = \int_{-\pi/2}^{\pi/2}\left(1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(\frac{3}{2}+2\cos 2\theta+\frac{1}{2}\cos 4\theta\right)d\theta = \left[\frac{3}{2}\theta+\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \left(\frac{3}{2}\cdot\frac{\pi}{2}+\sin\pi+\frac{1}{8}\cdot\sin 2\pi\right)-\left(-\frac{3}{2}\cdot\frac{\pi}{2}-\sin\pi-\frac{1}{8}\sin 2\pi\right)$$

$$= \frac{3\pi}{2}.$$

Example 35.

Compute the double integral $\iint\limits_R \sin\sqrt{x^2+y^2}\,dxdy$ by transforming to polar coordinates. R is the disk $x^2+y^2 \le \pi$.

Solution.

The region R is indicated in Figure 34.

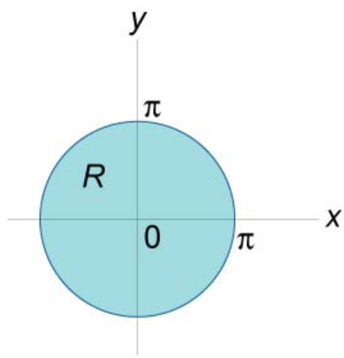

Figure 34.

The pullback S of the given region R can be written in the form $S = \{(r,\theta) \mid 0 \le r \le \pi, 0 \le \theta \le 2\pi\}$ (see Figure 35).

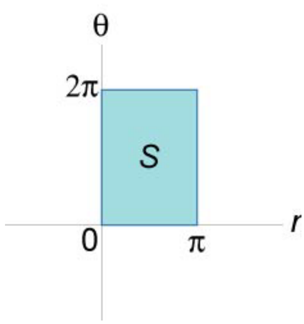

Figure 35.

The double integral in polar coordinates becomes

$$I = \iint_R \sin\sqrt{x^2 + y^2}\,dxdy = \iint_S \sin\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}\; rdrd\theta$$

$$= \iint_S r\sin r drd\theta = \int_0^{2\pi} d\theta \int_0^{\pi} r\sin r dr = 2\pi\int_0^{\pi} r\sin r dr\,.$$

We compute this integral using integration by parts:

$$\int_a^b udv = (uv)\Big|_a^b - \int_a^b vdu\,.$$

Let $u = r$, $dv = \sin r dr$. Then $du = dr$, $v = \int \sin r dr = -\cos r$.

Hence,

$$I = 2\pi\int_0^{\pi} r\sin r dr = 2\pi\left[(-r\cos r)\Big|_0^{\pi} - \int_0^{\pi}(-\cos r)dr\right]$$

$$= 2\pi\left[(-r\cos r)\Big|_0^{\pi} + \int_0^{\pi}\cos r dr\right]$$

$$= 2\pi\left[(-r\cos r)\Big|_0^{\pi} + (\sin r)\Big|_0^{\pi}\right]$$

$$= 2\pi(\sin r - r\cos r)\Big|_0^{\pi}$$

$$= 2\pi[(\sin\pi - \pi\cos\pi) - (\sin 0 - 0\cdot\cos 0)]$$

$$= 2\pi\cdot\pi = 2\pi^2\,.$$