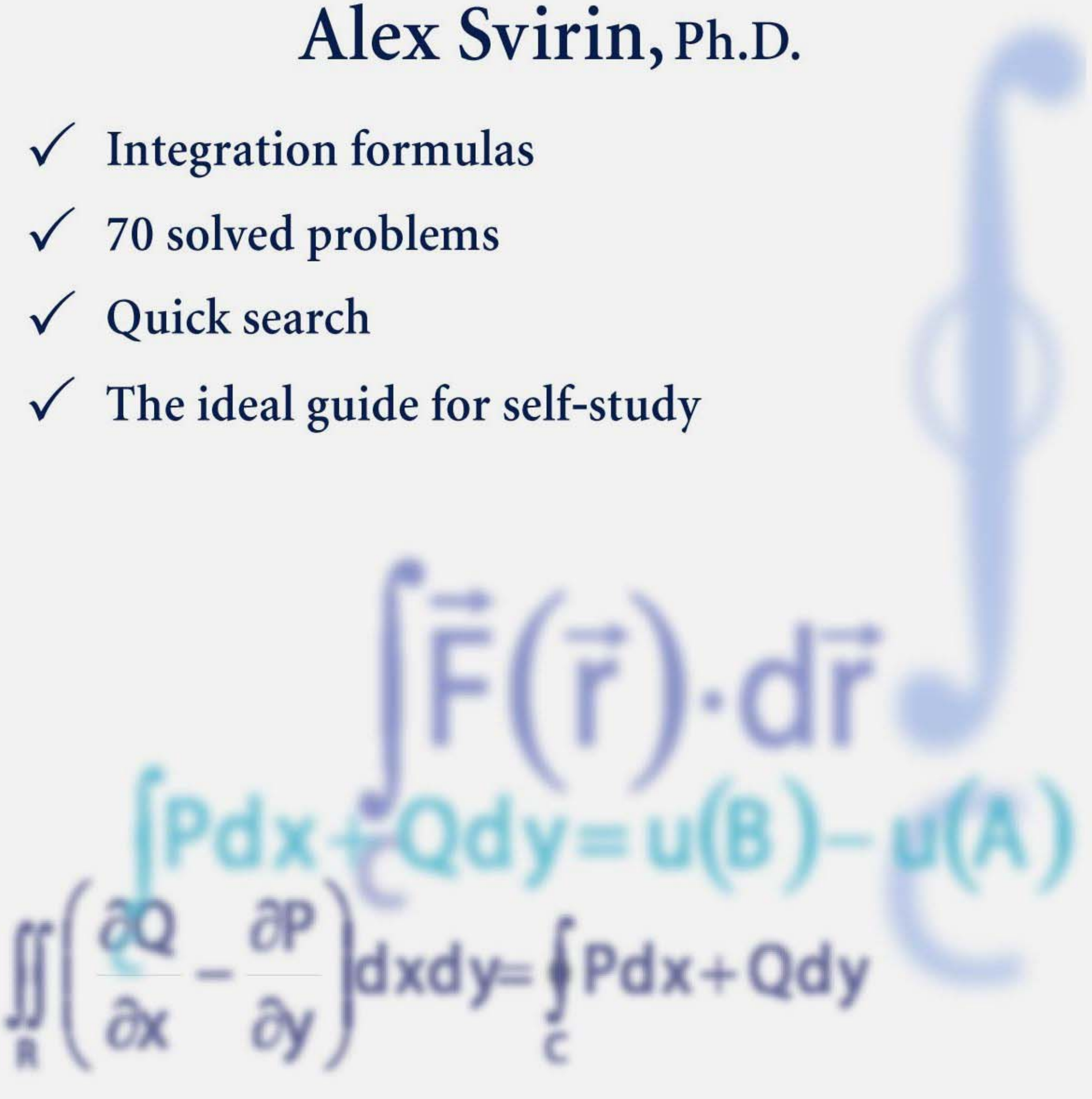
Calculus

Line Integrals

Alex Svirin, Ph.D.

- ✓ Integration formulas
- ✓ 70 solved problems
- ✓ Quick search
- ✓ The ideal guide for self-study



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Preface

This study guide is the most complete resource on line integrals. It contains 70 different problems solved step-by-step. If you take the time to work through this ebook, you will be a master of the topic. The areas covered are Line Integrals of Scalar Functions, Line integrals of Vector Fields, Green's Theorem, Path Independence of Line Integrals, and Applications of Line Integrals. Each of the chapters contains appropriate definitions and formulas followed by solved problems. Studying and solving these problems helps you cut study time and improve problem solving skills. The ebook is well suited for preparation before an exam.

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Chapter 3

Green's Theorem

Let R be a region in the xy-plane that is bounded by a closed, piecewise smooth curve C. Let

$$\vec{F} = P(x,y)\vec{i} + Q(x,y)\vec{j}$$

be a continuous vector function with continuous first partial derivatives $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ in a some domain containing R. Then

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_C Pdx + Qdy,$$

where the symbol $\oint_C$ indicates that the curve (contour) C is closed and integration is performed counterclockwise around C.

If $Q = x$, $P = -y$, Green's formula yields:

$$S = \iint_R dxdy = \frac{1}{2} \oint_C xdy - ydx,$$

where S is the area of the region R bounded by the contour C.

We can also write Green's Theorem in vector form. For this we introduce the Divergence and Curl of a vector field. Let

$$\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$$

be a vector field. Then the divergence of the field is defined as

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\,.$$

In terms of divergence and curl, Green's Theorem can be written as

$$\iint_R \left(\operatorname{curl}\vec{F}\right)\cdot\vec{k}\,dxdy = \oint_C \vec{F}\cdot d\vec{r}\,.$$

Note that Green's Theorem is simply Stoke's Theorem applied to a 2-dimensional plane.

SOLVED PROBLEMS

Example 26.

Using Green's theorem, evaluate the line integral $\oint_C xydx + (x+y)dy$, where C is the curve bounding the unit disk R.

Solution.

According to Green's formula

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \oint_C Pdx + Qdy,$$

we identify

$P(x,y) = xy$, $Q(x,y) = x + y$,

so that

$$I = \oint_C xydx + (x+y)dy = \iint_R \left(\frac{\partial(x+y)}{\partial x} - \frac{\partial(xy)}{\partial y} \right) dxdy = \iint_R (1-x)dxdy.$$

Converting the double integral into polar coordinates, we have

$$I = \iint_R (1-x)dxdy = \int_0^{2\pi}\int_0^1 (1 - r\cos\theta)rdrd\theta = \int_0^{2\pi}\left[\int_0^1 \left(r - r^2\cos\theta\right)dr\right]d\theta$$

$$= \int_0^{2\pi}\left[\left.\left(\frac{r^2}{2} - \frac{r^3}{3}\cos\theta\right)\right|_{r=0}^{1}\right]d\theta = \int_0^{2\pi}\left(\frac{1}{2} - \frac{\cos\theta}{3}\right)d\theta = \left[\frac{\theta}{2} - \frac{\sin\theta}{3}\right]_0^{2\pi} = \pi.$$

Example 27.

Using Green's theorem, evaluate the line integral $\oint_C x^2ydx - xy^2dy$, where the curve C is the circle $x^2 + y^2 = a^2$ traversed in the counter clockwise direction (Figure 17).

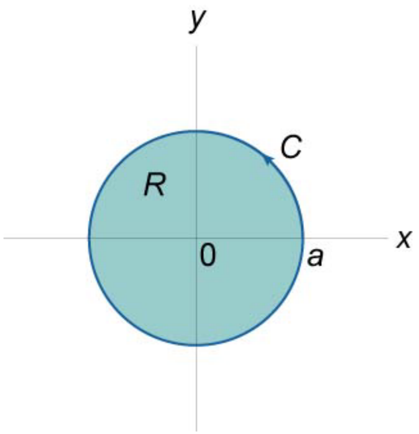

Figure 17.

Solution.

By Green's formula,

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \oint_C Pdx + Qdy .$$

We identify

$$P(x,y) = x^2 y,\ Q(x,y) = -xy^2,$$

$$\frac{\partial Q}{\partial x} = \frac{\partial\left(-xy^2\right)}{\partial x} = -y^2,\ \frac{\partial P}{\partial y} = \frac{\partial\left(x^2 y\right)}{\partial y} = x^2 .$$

Then

$$I = \oint_C x^2 ydx - xy^2 dy = \iint_R \left(-y^2 - x^2\right) dxdy = -\iint_R \left(x^2 + y^2\right) dxdy,$$

where R is the circle with radius a centered at the origin. Transforming to polar coordinates, we obtain

$$I = -\iint_R \left(x^2 + y^2\right) dxdy = -\int_0^{2\pi} d\theta \int_0^a \left(r^2 \cos^2\theta + r^2 \sin^2\theta\right) rdr$$

$$=-\int_0^{2\pi} d\theta \int_0^a r^3 dr = -2\pi \cdot \left(\frac{r^4}{4}\right)\Bigg|_0^a = -\frac{\pi a^4}{4}.$$

Example 28.

Using Green's formula, evaluate the line integral $\oint_C (x-y)dx+(x+y)dy$ where C is the circle $x^2+y^2=a^2$.

Solution.

According to Green's formula

$$\oint_C Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy,$$

we identify

$P=x-y$, $Q=x+y$, so that

$$\frac{\partial Q}{\partial x}=\frac{\partial(x+y)}{\partial x}=1,\ \frac{\partial P}{\partial y}=\frac{\partial(x-y)}{\partial y}=-1.$$

Then the line integral is

$$I=\oint_C (x-y)dx+(x+y)dy=\iint_R (1-(-1))dxdy=2\iint_R dxdy.$$

The double integral $\iint_R dxdy$ is equal to the area of the disk $x^2+y^2=a^2$, which is πa^2.

Then the answer is

$$I=2\iint_R dxdy=2\pi a^2.$$

Example 29.

Using Green's formula, evaluate the line integral

$\oint_C (x+y)dx-(x-y)dy$, where C is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (see Figure 18).

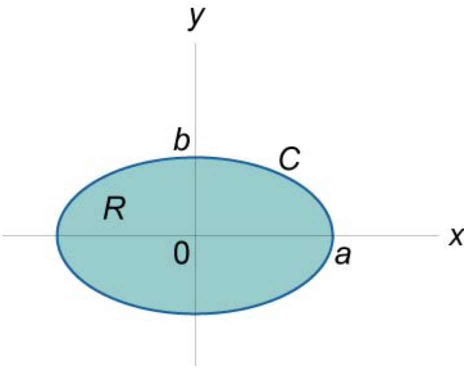

Figure 18.

Solution.

We use Green's formula

$$\oint_C Pdx+Qdy=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy.$$

Here

$$P=x+y,\ Q=-(x-y),\ \frac{\partial Q}{\partial x}=-1,\ \frac{\partial P}{\partial y}=1.$$

Hence,

$$I=\oint_C (x+y)dx-(x-y)dy=\iint_R(-1-1)dxdy=-2\iint_R dxdy.$$

The double integral $\iint_R dxdy$ is equal to the area of the ellipse, which is πab.

Then the line integral is

$$I=-2\iint_R dxdy=-2\pi ab.$$

Example 30.

Using Green’s formula, evaluate the line integral $\oint_C y^2dx+(x+y)^2dy$ along the contour of the triangle ABD with vertices $A(a,0)$, $B(a,a)$, $D(0,a)$ (Figure 19).

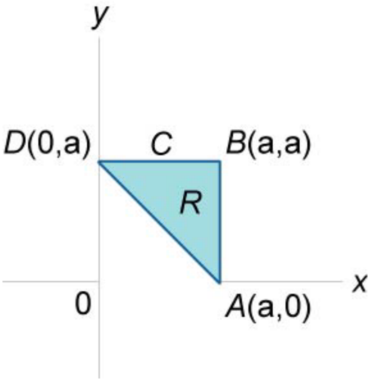

Figure 19.

Solution.

In the given line integral $P=y^2$, $Q=(x+y)^2$, so that

$$\frac{\partial Q}{\partial x}=\frac{\partial\left((x+y)^2\right)}{\partial x}=2(x+y),\ \frac{\partial P}{\partial y}=\frac{\partial\left(y^2\right)}{\partial y}=2y.$$

Then, by Green’s formula,

$$I=\oint_C y^2dx+(x+y)^2dy=\iint_R(2(x+y)-2y)dxdy=2\iint_R xdxdy.$$

It is easy to see that the equation of the line AD is $y=-x+a$. Hence, the latter double integral becomes

$$I = 2\iint_R x\,dx\,dy = 2\int_0^a \left[\int_{-x+a}^{a} dy\right] x\,dx = 2\int_0^a \left[y\Big|_{-x+a}^{a}\right] x\,dx$$

$$= 2\int_0^a (a-(-x+a))x\,dx = 2\int_0^a x^2\,dx = 2\left(\frac{x^3}{3}\right)\Bigg|_0^a = \frac{2a^3}{3}.$$

Example 31.

Using Green's theorem, evaluate the line integral $\oint_C (y-x^2)dx-(x+y^2)dy$, where the contour C encloses the sector of the circle with radius a lying in the first quadrant.

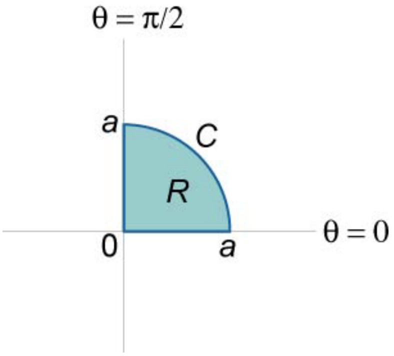

Figure 20.

Solution.

Applying Green's formula

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

we identify

$$P = y - x^2,\ Q = -(x+y^2),$$

$$\frac{\partial Q}{\partial x}=-\frac{\partial\left(x+y^2\right)}{\partial x}=-1,\ \frac{\partial P}{\partial y}=\frac{\partial\left(y-x^2\right)}{\partial y}=1.$$

Hence,

$$I=\oint_C\left(y-x^2\right)dx-\left(x+y^2\right)dy=\iint_R(-1-1)dxdy=-2\iint_R dxdy.$$

We transform this integral to polar coordinates. Then

$$I=-2\iint_R dxdy=-2\int_0^{\pi/2}d\theta\int_0^a rdr=-2\cdot\frac{\pi}{2}\cdot\left.\left(\frac{r^2}{2}\right)\right|_0^a=-\frac{\pi a^2}{2}.$$

Example 32.

Using Green's theorem, evaluate the line integral $\int_C\frac{dx-dy}{x+y}$, where C is the boundary of the square with the vertices A(1,0), B(0,1), D(-1,0), E(0,-1) (see Figure 21).

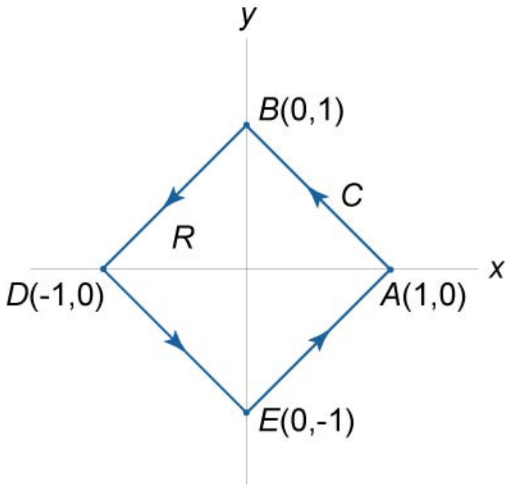

Figure 21.

Solution.

We use Green's theorem. Here

$$P=\frac{1}{x+y},\ Q=-\frac{1}{x+y},$$

$$\frac{\partial Q}{\partial x}=\frac{\partial\left(-\frac{1}{x+y}\right)}{\partial x}=\frac{1}{(x+y)^2},\ \frac{\partial P}{\partial y}=\frac{\partial\left(\frac{1}{x+y}\right)}{\partial x}=-\frac{1}{(x+y)^2}.$$

Hence,

$$I=\int_C\frac{dx-dy}{x+y}=\iint_R\left(\frac{1}{(x+y)^2}+\frac{1}{(x+y)^2}\right)dxdy=2\iint_R\frac{dxdy}{(x+y)^2}.$$

Find the equations of the sides of the square:

$AB:\ y=-x+1,$

$BD:\ y=x+1,$

$DE:\ y=-x-1,$

$EA:\ y=x-1.$

It is convenient to change variables. Let $u=y+x$, $v=y-x$.

In terms of u and v, we have

$y=-x+1, \quad \Rightarrow\ y+x=1 \text{ or } u=1,$

$y=x+1, \quad \Rightarrow\ y-x=1 \text{ or } v=1,$

$y=-x-1, \quad \Rightarrow\ y+x=-1 \text{ or } u=-1,$

$y=x-1, \quad \Rightarrow\ y-x=-1 \text{ or } v=-1.$

As seen, the pullback S of the initial region R is the square, shown in Figure 22.

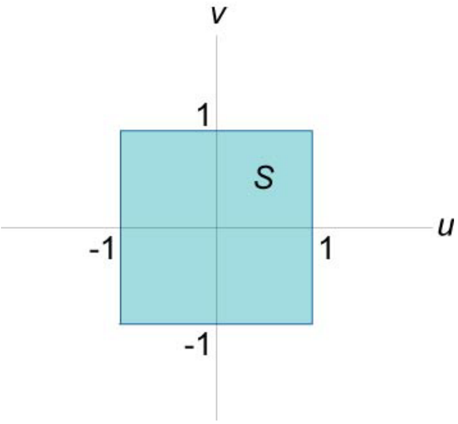

Figure 22.

Calculate the Jacobian of the transformation:

$$\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=\begin{vmatrix}\frac{\partial(y+x)}{\partial x} & \frac{\partial(y+x)}{\partial y}\\ \frac{\partial(y-x)}{\partial x} & \frac{\partial(y-x)}{\partial y}\end{vmatrix}=\begin{vmatrix}1 & 1\\ -1 & 1\end{vmatrix}=2\,.$$

Then the absolute value of the Jacobian is

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=\left|\left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1}\right|=\frac{1}{2}\,.$$

Hence,

$$dxdy=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv=\frac{1}{2}dudv\,,$$

so that the integral is

$$I=2\iint\limits_R\frac{dxdy}{(x+y)^2}=2\iint\limits_S\frac{\frac{1}{2}dudu}{u^2}=\iint\limits_S\frac{dudu}{u^2}=\int\limits_{-1}^{1}dv\int\limits_{-1}^{1}\frac{du}{u^2}$$

$$=v\Big|_{-1}^{1}\cdot\left(-\frac{1}{u}\right)\Big|_{-1}^{1}=(1-(-1))\cdot(-1-(-1))=-4\,.$$

Example 33.

Using Green's theorem, evaluate the line integral $\int\limits_C \sqrt{x^2+y^2}dx + y\left(xy + \ln\left(x+\sqrt{x^2+y^2}\right)\right)dy$, where C is the circle $x^2+x^2=a^2$.

Solution.

We identify

$$P=\sqrt{x^2+y^2},\ Q=y\left(xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right),$$

$$\frac{\partial Q}{\partial x}=\frac{\partial\left(xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right)}{\partial x}=y\left(y+\frac{1+\frac{2x}{2\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}\right)$$

$$=y\left(y+\frac{\frac{x+\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}}{x+\sqrt{x^2+y^2}}\right)=y\left(y+\frac{1}{\sqrt{x^2+y^2}}\right)=y^2+\frac{y}{\sqrt{x^2+y^2}},$$

$$\frac{\partial P}{\partial y}=\frac{\partial\sqrt{x^2+y^2}}{\partial y}=\frac{y}{\sqrt{x^2+y^2}}.$$

By Green's formula,

$$I=\int\limits_C\sqrt{x^2+y^2}dx+y\left(xy+\ln\left(x+\sqrt{x^2+y^2}\right)\right)dy$$

$$=\iint\limits_R\left(y^2+\frac{y}{\sqrt{x^2+y^2}}-\frac{y}{\sqrt{x^2+y^2}}\right)dxdy=\iint\limits_R y^2dxdy.$$

The region of integration R and the contour C are shown in Figure 23.

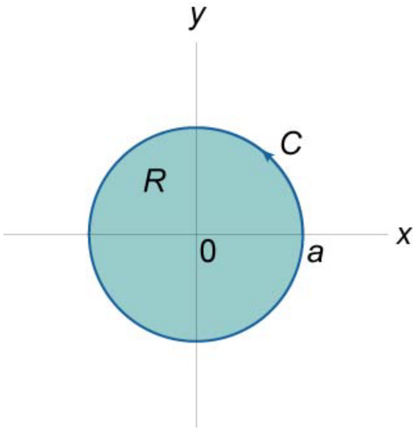

Figure 23.

It is convenient to transform the integral to polar coordinates.

$$I = \iint\limits_R y^2 dxdy = \int_0^{2\pi}\left[\int_0^a r^2 \sin^2\theta \cdot rdr\right]d\theta = \int_0^{2\pi}\sin^2\theta d\theta \int_0^a r^3 dr .$$

Here

$$\int_0^{2\pi}\sin^2\theta d\theta = \int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta = \frac{1}{2}\left[\theta - \frac{\sin\theta}{2}\right]\Bigg|_0^{2\pi} = \frac{1}{2}\cdot 2\pi = \pi ,$$

$$\int_0^a r^3 dr = \left(\frac{r^4}{4}\right)\Bigg|_0^a = \frac{a^4}{4} .$$

Hence,

$$I = \frac{\pi a^4}{4} .$$

Example 34.

Compute the area of the region R bounded by the astroid
$x = a\cos^3 t$, $y = a\sin^3 t$, $0 \le t \le 2\pi$.

Solution.

The area of the region R is given by the line integral

$$S=\frac{1}{2}\oint_C xdy-ydx\,.$$

In parametric form this formula is represented as

$$S=\frac{1}{2}\int_\alpha^\beta\left(x(t)\frac{dy}{dt}-y(t)\frac{dx}{dt}\right)dt\,.$$

Substituting the equations of the astroid, we obtain

$$\begin{aligned}
S&=\frac{1}{2}\int_\alpha^\beta\left(a\cos^3 t\cdot\frac{d\left(a\sin^3 t\right)}{dt}-a\sin^3 t\cdot\frac{d\left(a\cos^3 t\right)}{dt}\right)dt\\
&=\frac{1}{2}\int_0^{2\pi}\left(a\cos^3 t\cdot 3a\sin^2 t\cos t-a\sin^3 t\cdot 3a\cos^2 t(-\sin t)\right)dt\\
&=\frac{3a^2}{2}\int_0^{2\pi}\left(\cos^4 t\sin^2 t+\sin^4 t\cos^2 t\right)dt\\
&=\frac{3a^2}{8}\int_0^{2\pi}\left(4\sin^2 t\cos^2 t\left(\cos^2 t+\sin^2 t\right)\right)dt\\
&=\frac{3a^2}{8}\int_0^{2\pi}(\sin 2t)^2dt=\frac{3a^2}{8}\int_0^{2\pi}\frac{1-\cos 4t}{2}dt\\
&=\frac{3a^2}{16}\int_0^{2\pi}(1-\cos 4t)dt=\frac{3a^2}{16}\left[t-\frac{\sin 4t}{4}\right]\Bigg|_0^{2\pi}\\
&=\frac{3a^2}{16}\cdot 2\pi=\frac{3a^2\pi}{8}.
\end{aligned}$$

Example 35.

Verify Green's theorem when $\vec{F}(P,Q)=\left\langle x^2y^2,xy^2\right\rangle$ and the region of integration R is the disk of radius 2 with center at the origin.

Solution.

Calculate the line integral

$$I_1 = \oint_C P dx + Q dy = \int_C x^2 y^2 dx + xy^2 dy .$$

Using the parametric equation of the circle

$x = 2\cos t$, $y = 2\sin t$, $0 \le t \le 2\pi$,

we have

$$I_1 = \int_0^{2\pi} \left((2\cos t)^2 \cdot (2\sin t)^2 \cdot \frac{d(2\cos t)}{dt} + 2\cos t \cdot (2\sin t)^2 \cdot \frac{d(2\sin t)}{dt} \right) dt$$

$$= \int_0^{2\pi} \left(16\cos^2 t \sin^2 t \cdot (-2\sin t) + 8\cos t \sin^2 t \cdot 2\cos t \right) dt$$

$$= \int_0^{2\pi} \left(16\cos^2 t \sin^2 t (1 - 2\sin t) \right) dt$$

$$= 4\int_0^{2\pi} (\sin 2t)^2 (1 - \sin 2t) dt$$

$$= 4\int_0^{2\pi} \frac{1 - \cos 4t}{2} \cdot (1 - \sin 2t) dt$$

$$= 2\int_0^{2\pi} (1 - \cos 4t - \sin 2t + \sin 2t \cos 4t) dt$$

$$= 2\int_0^{2\pi} \left(1 - \cos 4t - \sin 2t + \frac{1}{2}(\sin 6t + \sin(-2t)) \right) dt .$$

We use here the trigonometric identity

$$\sin\alpha \cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)].$$

Then

$$I_1 = \int_0^{2\pi} (2 - 2\cos 4t - 3\sin 2t + \sin 6t) dt$$

$$=\left[2t-\frac{\sin 4t}{2}+\frac{3\cos 2t}{2}-\frac{\cos 6t}{6}\right]\Bigg|_0^{2\pi}=$$

$$=\left(4\pi+\frac{3}{2}-\frac{1}{6}\right)-\left(\frac{3}{2}-\frac{1}{6}\right)=4\pi\,.$$

Now we evaluate the double integral:

$$I_2=\iint_R\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy=\iint_R\left(\frac{\partial\left(xy^2\right)}{\partial x}-\frac{\partial\left(x^2y^2\right)}{\partial y}\right)dxdy$$

$$=\iint_R\left(y^2-2yx^2\right)dxdy\,.$$

In polar coordinates, the integral can be written as

$$I_2=\iint_R\left(y^2-2yx^2\right)dxdy=\int_0^{2\pi}\int_0^2\left((r\sin\theta)^2-2r\sin\theta(r\cos\theta)^2\right)rdrd\theta$$

$$=\int_0^{2\pi}\left[\int_0^2\left(r^2\sin^2\theta-2r^3\sin\theta\cos^2\theta\right)rdr\right]d\theta$$

$$=\int_0^{2\pi}\left[\int_0^2\left(r^3\sin^2\theta-2r^4\sin\theta\cos^2\theta\right)dr\right]d\theta$$

$$=\int_0^{2\pi}\left[\left(\frac{r^4}{4}\sin^2\theta-\frac{2r^5}{5}\sin\theta\cos^2\theta\right)\Bigg|_{r=0}^{2}\right]d\theta$$

$$=\int_0^{2\pi}\left(4\sin^2\theta-\frac{64}{5}\sin\theta\cos^2\theta\right)d\theta$$

$$=4\int_0^{2\pi}\sin^2\theta d\theta-\frac{64}{5}\int_0^{2\pi}\sin\theta\cos^2\theta d\theta$$

$$=4\int_0^{2\pi}\frac{1-\cos 2\theta}{2}d\theta+\frac{64}{5}\int_0^{2\pi}\cos^2\theta d(\cos\theta)$$

$$= 2\left(\theta - \frac{\sin 2\theta}{2}\right)\Bigg|_0^{2\pi} + \frac{64}{5}\left(\frac{\cos^3\theta}{3}\right)\Bigg|_0^{2\pi} = 4\pi\,.$$

As seen, $I_1 = I_2$.