# Calculus
# Surface Integrals

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# Preface

This ebook is intended for calculus students and instructors and gives a complete overview of surface integrals. It contains 50 completely worked problems with full solutions. The sample problems cover such topics as Surface Integrals of Scalar Functions, Surface Integrals of Vector Fields, The Divergence Theorem, Stoke's Theorem, and Applications of Surface Integrals. Each chapter begins with a brief statement of definition and theory accompanied by original problems and others modified from existing literature. Any problem or type of problems pertinent to the student's understanding of the subject is included. This study guide is well suited for preparation before an exam.

# Contents

# Chapter 4

# Stoke's Theorem

Let S be a smooth surface with a smooth bounding curve C. Then for any continuously differentiable vector function

$$\vec{F}(x,y,z)=\langle P(x,y,z),Q(x,y,z),R(x,y,z)\rangle$$

Stoke's Theorem states

$$\oint_C \vec{F}\cdot d\vec{r}=\iint_S \left(\nabla\times\vec{F}\right)\cdot d\vec{S},$$

where

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial x} & \frac{\partial}{\partial x}\\ P & Q & R\end{vmatrix}=\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i}+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{j}+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k}$$

is the curl of $\vec{F}$, also denoted $\operatorname{curl}\vec{F}$.

The symbol $\oint$ indicates that the line integral is taken over a closed curve.

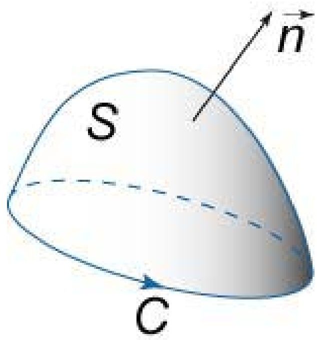


**Figure 14.**

We assume there is an orientation on both the surface and the curve that are related by the right hand rule. That is, if you were to walk around the curve in its preferred direction with your head pointing in the same direction as the normal vector $\vec{n}$ to the surface, then the surface would always be on your left (see Figure 14).

Stoke's Theorem relates line integrals of vector fields to surface integrals of vector fields.

In coordinate form Stoke's Theorem can be written as

$$\oint_C P dx + Q dy + R dz$$

$$= \iint_S \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy .$$

# SOLVED PROBLEMS

## Example 25.

Show that the line integral $\oint_C yzdx + xzdy + xydz$ is zero along any closed contour C.

## Solution.

Let S be a surface bounded by a closed curve C. Applying Stoke's formula, we identify that

$P = yz$, $Q = xz$, $R = xy$.

Then

$$\nabla \times \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k}$$

$$= (x - x)\vec{i} + (y - y)\vec{j} + (z - z)\vec{k}$$

$$= 0 \cdot \vec{i} + 0 \cdot \vec{j} + 0 \cdot \vec{k} = \vec{0}.$$

Hence

$$\oint_C yzdx + xzdy + xydz = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S \vec{0} \cdot d\vec{S} = 0,$$

so the statement is proved.

## Example 26.

Use Stoke's Theorem to evaluate the line integral

$\oint_C (y + 2z)dx + (x + 2z)dy + (x + 2y)dz$, where C is the curve formed by intersection of the sphere $x^2 + y^2 + z^2 = 1$ with the plane $x + 2y + 2z = 0$.

# Solution.

Let S be the circle cut by the sphere from the plane. The unit normal vector $\vec{n}$ is

$$\vec{n} = \frac{1\cdot\vec{i} + 2\cdot\vec{j} + 2\cdot\vec{k}}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}.$$

We identify that

$P = y + 2z$, $Q = x + 2z$, $R = x + 2y$.

Then the curl of $\vec{F}$ is

$$\nabla\times\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (2-2)\vec{i} + (2-1)\vec{j} + (1-1)\vec{k} = \vec{j}.$$

Using Stoke's Theorem, we have

$$I = \oint_C (y+2z)dx + (x+2z)dy + (x+2y)dz$$

$$= \iint_S \left(\nabla\times\vec{F}\right)\cdot d\vec{S} = \iint_S \left(\nabla\times\vec{F}\right)\cdot\vec{n}dS$$

$$= \iint_S \vec{j}\cdot\left(\frac{1}{3}\vec{i} + \frac{2}{3}\vec{j} + \frac{2}{3}\vec{k}\right)dS = \frac{2}{3}\iint_S dS.$$

Since the sphere $x^2 + y^2 + z^2 = 1$ is centered at the origin and the plane $x + 2y + 2z = 0$ also passes through the origin, the cross section is the circle of radius 1. Hence the integral is

$$I = \frac{2}{3}\iint_S dS = \frac{2}{3}\cdot\pi\cdot 1^2 = \frac{2\pi}{3}.$$

## Example 27.

Use Stoke's Theorem to evaluate the line integral $\oint_C (x+z)dx + (x-y)dy + xdz$, where C is the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$, $z = 1$ (Figure 15).

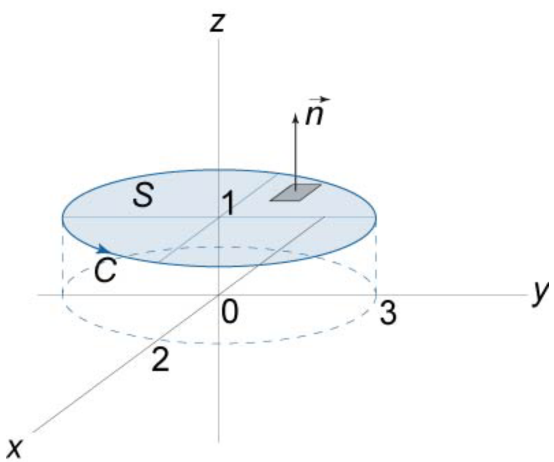


**Figure 15.**

## Solution.

Let the surface S be the part of the plane $z = 1$ bounded by the ellipse. Obviously that the unit normal vector is $\vec{n} = \vec{k}$. Since
$P = x + z$, $Q = x - y$, $R = x$,
we obtain

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (0-0)\vec{i} + (1-1)\vec{j} + (1-0)\vec{k} = \vec{k}.$$

Then by Stoke's Theorem,

$$\oint_C (x+z)dx + (x-y)dy + xdz$$
$$= \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS$$
$$= \iint_S \vec{k} \cdot \vec{k} dS = \iint_S dS .$$

The double integral $\iint_S dS$ is the area of the ellipse which is

$$\iint_S dS = \pi \cdot 2 \cdot 3 = 6\pi .$$

### Example 28.

Use Stoke's Theorem to evaluate the line integral $\oint_C y^3 dx - x^3 dy + z^3 dz$, where C is the intersection of the cylinder $x^2 + y^2 = a^2$ and the plane $x + y + z = b$.

## Solution.

We suppose that S is the part of the plane cut by the cylinder. The curve C is oriented counter-clockwise when viewed from the end of the normal vector $\vec{n}$ which has coordinates

$$\vec{n} = \frac{1 \cdot \vec{i} + 1 \cdot \vec{j} + 1 \cdot \vec{k}}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}} \vec{i} + \frac{1}{\sqrt{3}} \vec{j} + \frac{1}{\sqrt{3}} \vec{k} .$$

Since $P = y^3$, $Q = -x^3$, and $R = z^3$, we have

$$\nabla \times \vec{F} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$
$$= (0-0)\vec{i} + (0-0)\vec{j} + \left(-3x^2 - 3y^2\right)\vec{k} = -3\left(x^2 + y^2\right)\vec{k} .$$

Applying Stoke's Theorem, we find that

$$\oint_C y^3dx - x^3dy + z^3dz = \iint_S (\nabla\times\vec{F})\cdot d\vec{S}$$

$$= \iint_S (\nabla\times\vec{F})\cdot\vec{n}dS = \iint_S \left(-3(x^2+y^2)\vec{k}\right)\cdot\left(\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}\right)dS$$

$$= -\sqrt{3}\iint_S (x^2+y^2)dS = -\sqrt{3}a^2\iint_S dS.$$

The projection of the surface S onto the xy-plane is the circle $x^2+y^2=a^2$ of radius a. Therefore, representing the equation of the plane in the form $z=b-x-y$ and using the formula

$$\iint_S dS = \iint_{D(x,y)} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy$$ (see Chapter 1),

we obtain

$$I = -\sqrt{3}a^2\iint_S dS = -\sqrt{3}a^2 \iint_{D(x,y)} \sqrt{1+(-1)^2+(-1)^2}\,dxdy$$

$$= -3a^2\iint_{D(x,y)} dxdy = -3a^2\cdot\pi a^2 = -3\pi a^4.$$

## Example 29.

Use Stoke's Theorem to evaluate the line integral $\oint_C x^2y^3dx + 2dy + zdz$, where C is the circle in the xy-plane given by the equation $x^2+y^2=1$. The surface S is the upper hemisphere $x^2+y^2+z^2=1$.

# Solution.

We identify that $P=x^2y^3, Q=2, R=z$. Therefore, the curl of $\vec{F}$ is

$$\nabla\times\vec{F} = \left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (0-0)\vec{i} + (0-0)\vec{j} + \left(-3x^2y^2\right)\vec{k} = \left(-3x^2y^2\right)\vec{k}.$$

Since the equation of the upper hemisphere (at $z \ge 0$) is $z = \sqrt{1-x^2-y^2}$, the vector area element of the surface (oriented upwards) is

$$d\vec{S} = \left(-\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k}\right)dxdy$$

$$= \left(\frac{x}{\sqrt{1-x^2-y^2}}\vec{i} + \frac{x}{\sqrt{1-x^2-y^2}}\vec{j} + \vec{k}\right)dxdy.$$

(It is supposed that the contour C is traversed in the counter clockwise direction if looking from the positive z-axis).

The line integral can be found through the surface integral:

$$I = \oint_C x^2y^3dx + 2dy + zdz = \iint_S \left(\nabla \times \vec{F}\right)\cdot d\vec{S}$$

$$= \iint_{D(x,y)} \left(-3x^2y^2\right)\vec{k}\cdot\left(\frac{x}{\sqrt{1-x^2-y^2}}\vec{i} + \frac{x}{\sqrt{1-x^2-y^2}}\vec{j} + \vec{k}\right)dxdy$$

$$= -3\iint_{D(x,y)} x^2y^2dxdy.$$

By changing to polar coordinates, we obtain

$$I = -3\iint_{D(x,y)} x^2y^2dxdy = -3\iint_{D(r,\varphi)} r^2\cos^2\varphi\cdot r^2\sin^2\varphi\cdot rdrd\varphi$$

$$= -\frac{3}{4}\iint_{D(r,\varphi)} (2\sin\varphi\cos\varphi)^2r^5drd\varphi = -\frac{3}{4}\iint_{D(r,\varphi)} (\sin 2\varphi)^2r^5drd\varphi$$

$$= -\frac{3}{8}\iint_{D(r,\varphi)} (1-\cos 4\varphi)r^5drd\varphi.$$

The region of integration $D(r,\varphi)$ is the circle $x^2+y^2\le 1$. Therefore

$$I=-\frac{3}{8}\iint_{D(r,\varphi)}(1-\cos 4\varphi)r^5 drd\varphi=-\frac{3}{8}\int_0^{2\pi}(1-\cos 4\varphi)d\varphi\int_0^1 r^5 dr$$

$$=-\frac{3}{8}\left(\varphi-\frac{\sin 4\varphi}{4}\right)\Bigg|_{\varphi=0}^{2\pi}\cdot\left(\frac{r^6}{6}\right)\Bigg|_{r=0}^{1}=-\frac{3}{8}\cdot 2\pi\cdot\frac{1}{6}=-\frac{\pi}{8}.$$

## Example 30.

Use Stoke's Theorem to evaluate the line integral $\oint_C (z-y)dx+(x-z)dy+(y-x)dz$ along the triangle with the vertices $A(2,0,0)$, $B(0,2,0)$, and $D(0,0,2)$ (Figure 16).

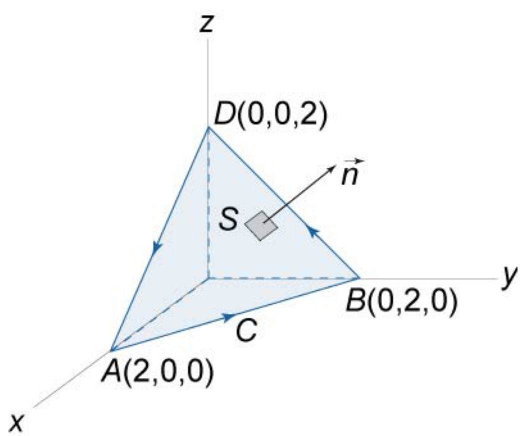


**Figure 16.**

# Solution.

We suppose that the surface S is the plane of the triangle ABD. Orientation of the surface S and the contour C are shown in Figure 16.

We first find the unit normal vector $\vec{n}$.

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle = \langle 0-2, 2-0, 0-0 \rangle = \langle -2, 2, 0 \rangle,$$

$$\overrightarrow{BD} = \langle x_D - x_B, y_D - y_B, z_D - z_B \rangle = \langle 0-0, 0-2, 2-0 \rangle = \langle 0, -2, 2 \rangle.$$

Then

$$\overrightarrow{AB} \times \overrightarrow{BD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 2 & 0 \\ 0 & -2 & 2 \end{vmatrix} = 4\vec{i} + 4\vec{j} + 4\vec{k}.$$

Hence

$$\vec{n} = \frac{\overrightarrow{AB} \times \overrightarrow{BD}}{\left|\overrightarrow{AB} \times \overrightarrow{BD}\right|} = \frac{4\vec{i} + 4\vec{j} + 4\vec{k}}{\sqrt{4^2 + 4^2 + 4^2}} = \frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}.$$

Since in our case $P = z - y$, $Q = x - z$, $R = y - x$. The curl of $\vec{F}$ is

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (1 - (-1))\vec{i} + (1 - (-1))\vec{j} + (1 - (-1))\vec{k} = 2\vec{i} + 2\vec{j} + 2\vec{k}.$$

By Stoke's formula,

$$I = \oint_C (z - y)dx + (x - z)dy + (y - x)dz$$

$$= \iint_S \left(\nabla \times \vec{F}\right) \cdot d\vec{S} = \iint_S \left(\nabla \times \vec{F}\right) \cdot \vec{n} dS$$

$$= \iint_S \left(2\vec{i} + 2\vec{j} + 2\vec{k}\right) \cdot \left(\frac{1}{\sqrt{3}}\vec{i} + \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k}\right) dS$$

$$= \frac{2}{\sqrt{3}} \iint_S (1+1+1) dS = 2\sqrt{3} \iint_S dS.$$

Here $\iint_S dS$ is the area of the triangle ABD which is equal to

$$A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{BD}\right| = \frac{1}{2} \cdot 4\sqrt{3} = 2\sqrt{3}.$$

Then the answer is

$$I = 2\sqrt{3} \iint_S dS = 2\sqrt{3} \cdot 2\sqrt{3} = 12.$$

## Example 31.

Use Stoke's Theorem to evaluate the line integral

$\oint_C \left(z^2 - y^2\right) dx + \left(x^2 - z^2\right) dy + \left(y^2 - x^2\right) dz$ where C is the curve formed by intersection of the paraboloid $z = 5 - x^2 - y^2$ with the plane $x + y + z = 1$ (see Figure 17).

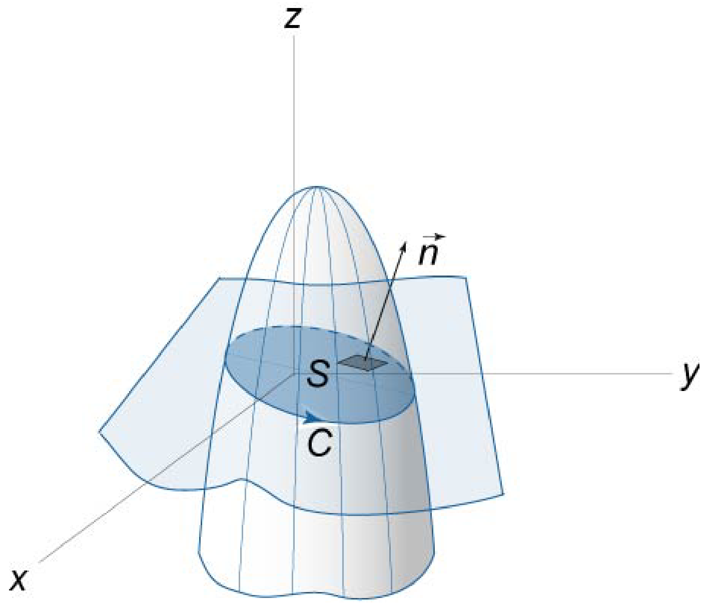


**Figure 17.**

## Solution.

Let S be the part of the plane $x+y+z=1$ cut by the paraboloid. Orientation of the surface S and the curve C is shown in Figure 17.

Find the normal vector $\vec{n}$ to the surface S. From the equation of the plane we obtain

$$\vec{n}=\frac{1\cdot\vec{i}+1\cdot\vec{j}+1\cdot\vec{k}}{\sqrt{1^2+1^2+1^2}}=\frac{1}{\sqrt{3}}\vec{i}+\frac{1}{\sqrt{3}}\vec{j}+\frac{1}{\sqrt{3}}\vec{k}.$$

Since

$P=z^2-y^2$, $Q=x^2-z^2$, $R=y^2-x^2$,

the curl of the vector field $\vec{F}(P,Q,R)$ is

$$\nabla \times \vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (2y + 2z)\vec{i} + (2z + 2x)\vec{j} + (2x + 2y)\vec{k}.$$

By Stoke's formula, we have

$$I = \oint_C (z^2 - y^2)dx + (x^2 - z^2)dy + (y^2 - x^2)dz$$

$$= \iint_S (\nabla \times \vec{F})\cdot d\vec{S} = \iint_S (\nabla \times \vec{F})\cdot \vec{n}dS$$

$$= \frac{2}{\sqrt{3}}\iint_S (y + z + z + x + x + y)dS = \frac{4}{\sqrt{3}}\iint_S (x + y + z)dS.$$

Since $x + y + z = 1$, the integral becomes

$$I = \frac{4}{\sqrt{3}}\iint_S dS.$$

To complete the calculation, we must evaluate the double integral $\iint_S dS$, i.e. the area of the surface S.

The explicit equation of the plane is $z = 1 - x - y$. Therefore, using the formula

$$\iint_S dS = \iint_{D(x,y)} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dxdy$$

(see Chapter 1), where $D(x,y)$ is projection of S onto the xy-plane, we have

$$I = \frac{4}{\sqrt{3}}\iint_S dS = \frac{4}{\sqrt{3}}\iint_{D(x,y)} \sqrt{1 + (-1)^2 + (-1)^2}\, dxdy = 4\iint_{D(x,y)} dxdy.$$

Determine the region of integration $D(x,y)$. Solving the system of the equations $\begin{cases} x+y+z=1 \\ z=5-x^2-y^2 \end{cases}$, we obtain

$$x+y+5-x^2-y^2=1,$$
$$x^2+y^2-x-y=4,$$
$$\left(x^2-x+\frac{1}{4}\right)+\left(y^2-y+\frac{1}{4}\right)=\frac{9}{2},$$
$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2=\left(\frac{3}{\sqrt{2}}\right)^2.$$

Thus we see that the region $D(x,y)$ is the circle of radius $R=\frac{3}{\sqrt{2}}$ centered at $\left(\frac{1}{2},\frac{1}{2}\right)$. Then the area of the region $D(x,y)$ is

$$\iint\limits_{D(x,y)} dxdy=\pi\left(\frac{3}{\sqrt{2}}\right)^2=\frac{9\pi}{2}.$$

Hence the initial integral is

$$I=4\cdot\frac{9\pi}{2}=18\pi.$$

## Example 32.

Verify that Stoke's Theorem is true for the vector field $\vec{F}(x,y,z)=z\vec{i}+x\vec{j}+y\vec{k}$. The surface S is the rectangular with the vertices $A(0,2,0)$, $B(0,0,2)$, $D(2,0,2)$, $E(2,2,0)$ (Figure 18).

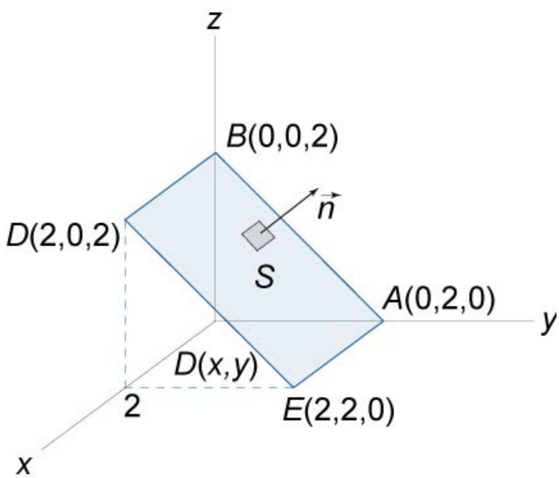


**Figure 18.**

## Solution.

Using Stoke's formula

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \left(\nabla\times\vec{F}\right)\cdot d\vec{S}\ ,$$

calculate first the surface integral in the right-hand side.

The curl of $\vec{F}$ is

$$\nabla\times\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= \left(\frac{\partial y}{\partial y} - \frac{\partial x}{\partial z}\right)\vec{i} + \left(\frac{\partial z}{\partial z} - \frac{\partial y}{\partial x}\right)\vec{j} + \left(\frac{\partial x}{\partial x} - \frac{\partial z}{\partial y}\right)\vec{k}$$

$$= \vec{i} + \vec{j} + \vec{k}\,.$$

We can express the equation of the plane ABDE (i.e. the surface S) explicitly.
The normal vector $\vec{n}$ can be found as the vector product:

$$\vec{n} = \frac{\overrightarrow{AB} \times \overrightarrow{BC}}{\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|}.$$

Here

$\overrightarrow{AB} = \langle 0-0, 0-2, 2-0 \rangle = \langle 0, -2, 2 \rangle$,

$\overrightarrow{BC} = \langle 2-0, 0-0, 2-2 \rangle = \langle 2, 0, 0 \rangle$.

Then

$$\overrightarrow{AB} \times \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & -2 & 2 \\ 2 & 0 & 0 \end{vmatrix} = 4\vec{j} + 4\vec{k}.$$

The unit normal vector $\vec{n}$ is

$$\vec{n} = \frac{\overrightarrow{AB} \times \overrightarrow{BC}}{\left|\overrightarrow{AB} \times \overrightarrow{BC}\right|} = \frac{4\vec{j} + 4\vec{k}}{\sqrt{32}} = \frac{1}{\sqrt{2}}\vec{j} + \frac{1}{\sqrt{2}}\vec{k}.$$

Hence the plane ABDE is defined by the equation

$(x - x_A)n_x + (y - y_A)n_y + (z - z_A)n_z = 0$,

where $x_A$, $y_A$, $z_A$ are the coordinates of the point $A(0,2,0)$, and $n_x$, $n_y$, $n_z$ are the components of the normal vector $\vec{n}$.

As a result, we have

$$(x-0)\cdot 0 + (y-2)\cdot\frac{1}{\sqrt{2}} + (z-0)\cdot\frac{1}{\sqrt{2}} = 0$$

or

$z = 2 - y$.

The surface integral becomes

$$\iint\limits_S \left(\nabla \times \vec{F}\right)\cdot d\vec{S} = \iint\limits_{D(x,y)} \left(\vec{i} + \vec{j} + \vec{k}\right)\cdot\left(-\frac{\partial z}{\partial x}\vec{i} - \frac{\partial z}{\partial y}\vec{j} + \vec{k}\right) dxdy$$

$$= \iint\limits_{D(x,y)} \left(\vec{i} + \vec{j} + \vec{k}\right)\cdot\left(0\cdot\vec{i} - (-1)\cdot\vec{j} + \vec{k}\right)dxdy$$

$$= \iint\limits_{D(x,y)} (1+1)dxdy = 2\iint\limits_{D(x,y)} dxdy = 2\int_0^2 dx\int_0^2 dy = 8\,.$$

Here the region $D(x, y)$ is the projection of the surface S onto the xy-plane.

Evaluate now the line integral $\oint\limits_C \vec{F}\cdot d\vec{r}$ along the closed contour ABDE.

The complete line integral $\oint\limits_C \vec{F}\cdot d\vec{r}$ can be represented in the form:

$$\oint\limits_C \vec{F}\cdot d\vec{r} = \oint\limits_{AB} \vec{F}\cdot d\vec{r} + \oint\limits_{BD} \vec{F}\cdot d\vec{r} + \oint\limits_{DE} \vec{F}\cdot d\vec{r} + \oint\limits_{EA} \vec{F}\cdot d\vec{r}\,.$$

Calculate the four line integrals separately.

1) The equation of the segment line AB in parametric form is

$$\frac{x - x_A}{x_B - x_A} = \frac{y - y_A}{y_B - y_A} = \frac{z - z_A}{z_B - z_A} = t\,,$$

$$\frac{x-0}{0-0} = \frac{y-2}{0-2} = \frac{z-0}{2-0} = t\,,$$

$$\frac{x}{0} = \frac{y-2}{-2} = \frac{z}{2} = t\,,$$

or

$$\begin{cases} x = 0 \\ y = 2 - 2t\,. \\ z = 2t \end{cases}$$

Hence

$\vec{r}(t) = \langle 0, 2-2t, 2t\rangle$ where t ranges from 0 to 1.

Then

$$\frac{d\vec{r}}{dt}=\langle 0,-2,2\rangle .$$

The line integral on AB is

$$I_1=\oint_{AB}\vec{F}\cdot d\vec{r}=\int_0^1\left(\vec{F}(x(t),y(t),z(t))\cdot\frac{d\vec{r}}{dt}\right)dt$$

$$=\int_0^1\left(2t\cdot\vec{i}+0\cdot\vec{j}+(2-2t)\cdot\vec{k}\right)\cdot\left(0\cdot\vec{i}-2\cdot\vec{j}+2\cdot\vec{k}\right)dt$$

$$=\int_0^1 2(2-2t)dt=4\int_0^1(1-t)dt=4\left(t-\frac{t^2}{2}\right)\Bigg|_0^1=2 .$$

Similarly, we find the other line integrals.

2) The segment line BD has the equation:

$$\frac{x-x_B}{x_D-x_B}=\frac{y-y_B}{y_D-y_B}=\frac{z-z_B}{z_D-z_B}=t ,$$

$$\frac{x-0}{2-0}=\frac{y-0}{0-0}=\frac{z-2}{2-2}=t ,$$

$$\frac{x}{2}=\frac{y}{0}=\frac{z-2}{0}=t ,$$

or

$$\begin{cases}x=2t\\ y=0\\ z=2\end{cases} .$$

Hence

$\vec{r}(t)=\langle 2t,0,2\rangle$ where t again ranges from 0 to 1.

$$\frac{d\vec{r}}{dt}=\langle 2,0,0\rangle ,$$

so that the line integral on BD is

$$I_2 = \oint_{BD} \vec{F} \cdot d\vec{r} = \int_0^1 \left(2 \cdot \vec{i} + 2t \cdot \vec{j} + 0 \cdot \vec{k}\right) \cdot \left(2 \cdot \vec{i} + 0 \cdot \vec{j} + 0 \cdot \vec{k}\right) dt$$

$$= \int_0^1 4dt = 4 .$$

3) The segment line DE is defined as

$$\frac{x - x_D}{x_E - x_D} = \frac{y - y_D}{y_E - y_D} = \frac{z - z_D}{z_E - z_D} = t ,$$

$$\frac{x - 2}{2 - 2} = \frac{y - 0}{2 - 0} = \frac{z - 2}{0 - 2} = t ,$$

$$\frac{x - 2}{0} = \frac{y}{2} = \frac{z - 2}{-2} = t ,$$

or

$$\begin{cases} x = 2 \\ y = 2t \\ z = 2 - 2t \end{cases} .$$

Then

$\vec{r}(t) = \langle 2, 2t, 2 - 2t \rangle$ where $t \in [0,1]$

and

$\dfrac{d\vec{r}}{dt} = \langle 0, 2, -2 \rangle$, so that

$$I_3 = \oint_{DE} \vec{F} \cdot d\vec{r} = \int_0^1 \left((2 - 2t) \cdot \vec{i} + 2 \cdot \vec{j} + 2t \cdot \vec{k}\right) \cdot \left(0 \cdot \vec{i} + 2 \cdot \vec{j} - 2 \cdot \vec{k}\right) dt$$

$$= \int_0^1 (4 - 4t) dt = 4 \int_0^1 (1 - t) dt = 4 \left( t - \frac{t^2}{2} \right) \Bigg|_0^1 = 2 .$$

4) The segment line EA is given by

$$\frac{x - x_E}{x_A - x_E} = \frac{y - y_E}{y_A - y_E} = \frac{z - z_E}{z_A - z_E} = t ,$$

$$\frac{x-2}{0-2}=\frac{y-2}{2-2}=\frac{z-0}{0-0}=t,$$

$$\frac{x-2}{-2}=\frac{y-2}{0}=\frac{z}{0}=t,$$

$$\begin{cases} x=2-2t \\ y=2 \\ z=0 \end{cases}.$$

Then

$\vec{r}(t)=\langle 2-2t, 2, 0\rangle$, $t\in[0,1]$,

and

$$\frac{d\vec{r}}{dt}=\langle -2, 0, 0\rangle.$$

The integral $I_4=\oint_{EA}\vec{F}\cdot d\vec{r}$ is

$$I_4=\oint_{EA}\vec{F}\cdot d\vec{r}=\int_0^1\left(0\cdot\vec{i}+(2-2t)\cdot\vec{j}+2\cdot\vec{k}\right)\cdot\left(-2\cdot\vec{i}+0\cdot\vec{j}+0\cdot\vec{k}\right)dt=0.$$

The complete line integral along the closed contour C is
$I=I_1+I_2+I_3+I_4=2+4+2+0=8$.

Thus, Stoke's theorem is verified.