

Calculus

Fourier Series

Alex Svirin, Ph.D.

- ✓ Formulas and definitions
- ✓ 60 solved problems
- ✓ Quick search
- ✓ The ideal guide for self-study

$$a_0 = \frac{1}{L} \int_a^b f(x) dx$$

$$\cos n\pi = (-1)^n$$

Calculus

Fourier Series

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Preface

This ebook contains a selection of 60 problems on Fourier series with full solutions. The problems cover the following topics: Definition of Fourier Series and Typical Examples, Fourier Series of Functions with an Arbitrary Period, Even and Odd Extensions, Complex Form, Convergence of Fourier Series, Bessel's Inequality and Parseval's Theorem, Differentiation and Integration of Fourier Series, Orthogonal Polynomials and Generalized Fourier Series. Each of the chapters includes appropriate definitions and formulas followed by solved problems listed in order of increasing difficulty. For students this ebook is a valuable complement to textbooks, for lecturers teaching calculus, a helpful reference.

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Chapter 2

Fourier Series of Functions with an Arbitrary Period

Fourier Series Expansion on the Interval [-L, L]

In this chapter we consider Fourier series for functions which are L-periodic. Assume that the function $f(x)$ is piecewise continuous on the interval $[-L, L]$. Using the substitution

$$x = \frac{Ly}{\pi} \quad (-\pi \leq x \leq \pi),$$

we obtain the function

$$F(y) = f\left(\frac{Ly}{\pi}\right),$$

which is defined and integrable on $[-\pi, \pi]$. According to the previous chapter, we can write the Fourier series for $F(y)$:

$$F(y) = f\left(\frac{Ly}{\pi}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos ny + b_n \sin ny).$$

The Fourier coefficients for the function are given by

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) dy$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \cos ny dy = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right) \cos ny dy$$

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$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \sin ny dy = \frac{1}{\pi} \int_{-\pi}^{\pi} f\left(\frac{Ly}{\pi}\right) \sin ny dy,$$

$$n = 1, 2, 3, \dots$$

Returning to the initial variables, i.e. setting $y = \frac{\pi x}{L}$, we obtain

the following trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx.$$

Fourier Series Expansion on the Interval [a, b]

If the function $f(x)$ is defined on the interval $[a, b]$, then its Fourier series representation is given by the same formula

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right),$$

where $L = \frac{b-a}{2}$ and Fourier coefficients are computed as follows:

$$a_0 = \frac{1}{L} \int_a^b f(x) dx,$$

$$a_n = \frac{1}{L} \int_a^b f(x) \cos \frac{n\pi x}{L} dx,$$

$$b_n = \frac{1}{L} \int_a^b f(x) \sin \frac{n\pi x}{L} dx.$$

$$n = 1, 2, 3, \dots$$

Even and Odd Functions

The Fourier series expansion of an **even** function $f(x)$ defined on the interval $[-L, L]$ has the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_0 = \frac{2}{L} \int_0^L f(x) dx,$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx.$$

The Fourier series expansion of an **odd** function $f(x)$ defined on the interval $[-L, L]$ is given by the formula

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

SOLVED PROBLEMS

Example 16.

Find the Fourier series of the function

$$f(x) = \begin{cases} A, & \text{if } 0 \leq x \leq L \\ 0, & \text{if } L < x \leq 2L \end{cases}.$$

Solution.

Determine the Fourier coefficients.

$$a_0 = \frac{1}{L} \int_a^b f(x) dx = \frac{1}{L} \int_0^L A dx = A,$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_a^b f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{L} \int_0^L A \cos \frac{n\pi x}{L} dx \\ &= \frac{A}{L} \left(\frac{L}{n\pi} \sin \frac{n\pi x}{L} \right) \Big|_0^L = \frac{A}{n\pi} (\sin n\pi - \sin 0) = 0, \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{L} \int_a^b f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{L} \int_0^L A \sin \frac{n\pi x}{L} dx \\ &= \frac{A}{L} \left(-\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right) \Big|_0^L = \frac{A}{n\pi} (-\cos n\pi + \cos 0) \\ &= \frac{A}{n\pi} [1 - (-1)^n] = \frac{A}{n\pi} [1 + (-1)^{n+1}]. \end{aligned}$$

For even $n = 2k$, $k = 1, 2, 3, \dots$,

$$b_{2k} = \frac{A}{2k\pi} [1 + (-1)^{2k+1}] = 0.$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

For odd $n = 2k - 1$, $k = 1, 2, 3, \dots$,

$$b_{2k-1} = \frac{A}{(2k-1)\pi} [1 + (-1)^{2k}] = \frac{2A}{(2k-1)\pi}.$$

Hence,

$$f(x) = \frac{A}{2} + \frac{2A}{\pi} \sum_{k=1}^{\infty} \frac{1}{2k-1} \sin\left(\frac{2k-1}{L} \pi x\right).$$

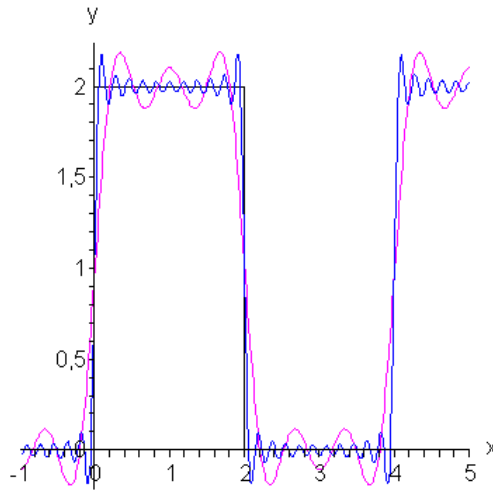


Figure 17. Fourier series for step function, $A = 2$, $L = 2$, $n = 2$, $n = 10$.

Example 17.

Find the Fourier series of the function

$$f(x) = \begin{cases} A, & \text{if } 0 \leq x \leq a \\ -A, & \text{if } a < x \leq b \end{cases}.$$

Solution.

For the given interval $L = \frac{b}{2}$. Then

$$a_0 = \frac{1}{L} \int_0^b f(x) dx = \frac{2}{b} \left[\int_0^a A dx - \int_a^b A dx \right] = \frac{2A}{b} (a - b + a) = \frac{2A(2a - b)}{b}.$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_0^b f(x) \cos \frac{n\pi x}{L} dx \\ &= \frac{2}{b} \left[\int_0^a A \cos \frac{2n\pi x}{b} dx - \int_a^b A \cos \frac{2n\pi x}{b} dx \right] \\ &= \frac{2A}{b} \cdot \frac{b}{2n\pi} \left[\left(\sin \frac{2n\pi x}{b} \right) \Big|_0^a - \left(\sin \frac{2n\pi x}{b} \right) \Big|_a^b \right] \\ &= \frac{A}{n\pi} \left[\sin \frac{2n\pi a}{b} - 0 - \sin 2n\pi + \sin \frac{2n\pi a}{b} \right] \\ &= \frac{2A}{n\pi} \sin \frac{2n\pi a}{b}. \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{L} \int_0^b f(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{2}{b} \left[\int_0^a A \sin \frac{2n\pi x}{b} dx - \int_a^b A \sin \frac{2n\pi x}{b} dx \right] \\ &= -\frac{2A}{b} \cdot \frac{b}{2n\pi} \left[\left(\cos \frac{2n\pi x}{b} \right) \Big|_0^a - \left(\cos \frac{2n\pi x}{b} \right) \Big|_a^b \right] \\ &= -\frac{A}{n\pi} \left[\cos \frac{2n\pi a}{b} - 1 - \left(\cos 2n\pi - \cos \frac{2n\pi a}{b} \right) \right] \end{aligned}$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

$$= \frac{2A}{n\pi} \left[1 - \cos \frac{2n\pi a}{b} \right].$$

Thus,

$$f(x) = \frac{A(2a-b)}{b} + \frac{2A}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \sin \frac{2n\pi a}{b} \cos \frac{2n\pi x}{b} + \left(1 - \cos \frac{2n\pi a}{b} \right) \sin \frac{2n\pi x}{b} \right\}.$$

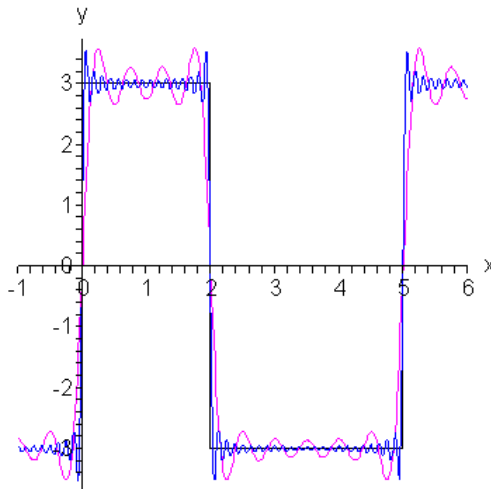


Figure 18. $A = 3$, $a=2$, $b = 5$, $n = 10$, $n = 40$.

Example 18.

Find the Fourier series of the function

$$f(x) = \begin{cases} 1, & \text{if } -1 \leq x \leq 1 \\ 0, & \text{if } 1 < x \leq 3 \end{cases}.$$

Solution.

Here $L = 2$. Hence,

$$a_0 = \frac{1}{L} \int_a^b f(x) dx = \frac{1}{2} \int_{-1}^3 f(x) dx = \frac{1}{2} \int_{-1}^1 dx = 1.$$

Coefficients a_n are

$$\begin{aligned} a_n &= \frac{1}{L} \int_a^b f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{2} \int_{-1}^1 \cos \frac{n\pi x}{2} dx = \frac{1}{2} \cdot \frac{2}{n\pi} \cdot \left(\sin \frac{n\pi x}{2} \right) \Big|_{-1}^1 \\ &= \frac{1}{n\pi} \left[\sin \frac{n\pi}{2} - \sin \left(-\frac{n\pi}{2} \right) \right] = \frac{2}{n\pi} \sin \frac{n\pi}{2}. \end{aligned}$$

For even $n = 2k$, $k = 1, 2, 3, \dots$,

$$\sin \frac{2k\pi}{2} = \sin k\pi = 0.$$

For odd $n = 2k + 1$, $k = 0, 1, 2, 3, \dots$,

$$\sin \frac{(2k+1)\pi}{2} = (-1)^k.$$

Therefore,

$$a_{2k+1} = \frac{2}{\pi} \cdot \frac{(-1)^k}{2k+1}, \quad k = 0, 1, 2, 3, \dots$$

Similarly, we compute b_n :

$$\begin{aligned} b_n &= \frac{1}{L} \int_a^b f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{2} \int_{-1}^1 \sin \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \cdot \frac{2}{n\pi} \cdot \left(-\cos \frac{n\pi x}{2} \right) \Big|_{-1}^1 = \frac{1}{n\pi} \left[-\cos \frac{n\pi}{2} + \cos \left(-\frac{n\pi}{2} \right) \right] = 0. \end{aligned}$$

This result is obvious since the given function is odd on the interval $[-1, 3]$.

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Thus,

$$f(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} \cos\left(\frac{2k+1}{2} \pi x\right).$$

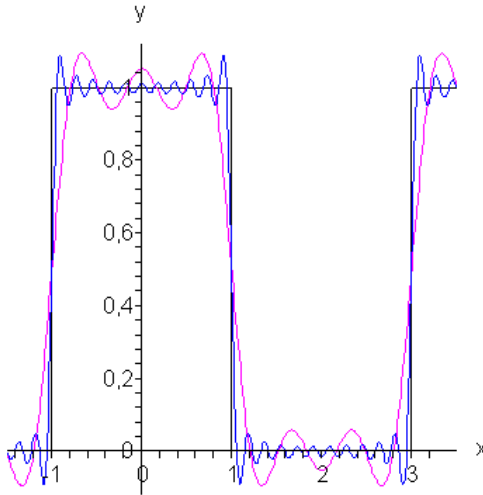


Figure 19. Fourier series for step function, $n = 2$, $n = 10$.

Example 19.

Find the Fourier series of the function

$$f(x) = \begin{cases} 0, & \text{if } -1 \leq x \leq 0 \\ x, & \text{if } 0 < x \leq 1 \end{cases}.$$

Solution.

Here $L = 1$. Then we can write

$$a_0 = \frac{1}{L} \int_a^b f(x) dx = \int_{-1}^1 f(x) dx = \int_0^1 x dx = \left(\frac{x^2}{2} \right) \Big|_0^1 = \frac{1}{2}.$$

Calculate a_n .

$$\begin{aligned}
 a_n &= \frac{1}{L} \int_a^b f(x) \cos \frac{n\pi x}{L} dx = \int_0^1 x \cos(n\pi x) dx \\
 &= \left(\frac{1}{n\pi} x \sin(n\pi x) \right) \Big|_0^1 - \frac{1}{n\pi} \int_0^1 \sin(n\pi x) dx \\
 &= \frac{1}{n\pi} \left[(x \sin n\pi x) \Big|_0^1 + \left(\frac{\cos n\pi x}{n\pi} \right) \Big|_0^1 \right] \\
 &= \frac{1}{n\pi} \left[\sin n\pi + \frac{\cos n\pi}{n\pi} - \frac{1}{n\pi} \right] \\
 &= \frac{1}{n^2 \pi^2} [\cos n\pi - 1] = \frac{1}{n^2 \pi^2} [(-1)^n - 1].
 \end{aligned}$$

Determine the coefficients b_n .

$$\begin{aligned}
 b_n &= \frac{1}{L} \int_a^b f(x) \sin \frac{n\pi x}{L} dx = \int_0^1 x \sin(n\pi x) dx \\
 &= \left(-\frac{1}{n\pi} x \cos(n\pi x) \right) \Big|_0^1 + \frac{1}{n\pi} \int_0^1 \cos(n\pi x) dx \\
 &= \frac{1}{n\pi} \left[(-x \cos n\pi x) \Big|_0^1 + \left(\frac{\sin n\pi x}{n\pi} \right) \Big|_0^1 \right] \\
 &= \frac{1}{n\pi} \left[-\cos n\pi + \frac{\sin n\pi}{n\pi} \right] = \frac{(-1)^{n+1}}{n\pi}.
 \end{aligned}$$

As a result, we obtain (Figure 20):

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left[\frac{((-1)^n - 1)}{n^2 \pi^2} \cos n\pi x + \frac{(-1)^{n+1}}{n\pi} \sin n\pi x \right].$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

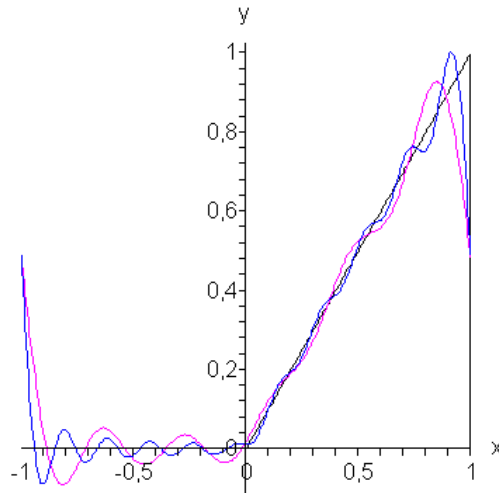


Figure 20. $n = 5$, $n = 10$.

Example 20.

Find the Fourier series of the function $f(x) = |\cos x|$.

Solution.

This function is even and has period π ($L = \frac{\pi}{2}$). As a result, we

obtain $b_n = 0$.

Compute the coefficients a_0 and a_n .

$$a_0 = \frac{2}{L} \int_0^L f(x) dx = \frac{4}{\pi} \int_0^{\pi/2} \cos x dx = \frac{4}{\pi} \cdot (\sin x) \Big|_0^{\pi/2} = \frac{4}{\pi}.$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{4}{\pi} \int_0^{\pi/2} \cos x \cos 2nx dx$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

$$\begin{aligned} &= \frac{2}{\pi} \int_0^{\pi/2} [\cos(2nx - x) + \cos(2nx + x)] dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} [\cos(2n-1)x + \cos(2n+1)x] dx \\ &= \frac{2}{\pi} \left[\frac{\sin(2n-1)x}{2n-1} + \frac{\sin(2n+1)x}{2n+1} \right]_0^{\pi/2} \\ &= \frac{2}{\pi} \left[\frac{\sin(2n-1)\frac{\pi}{2}}{2n-1} + \frac{\sin(2n+1)\frac{\pi}{2}}{2n+1} \right]. \end{aligned}$$

Since $\sin(2n-1)\frac{\pi}{2} = (-1)^{n+1}$ and $\sin(2n+1)\frac{\pi}{2} = (-1)^n$,

we have

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\frac{(-1)^{n+1}}{2n-1} + \frac{(-1)^n}{2n+1} \right] = \frac{2}{\pi} (-1)^{n+1} \cdot \frac{2n+1 - (2n-1)}{4n^2 - 1} \\ &= \frac{4}{\pi} \cdot \frac{(-1)^{n+1}}{4n^2 - 1}, \text{ where } n = 1, 2, 3, \dots \end{aligned}$$

Hence,

$$F(x) = \frac{x^2 - 2\pi x}{4}.$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

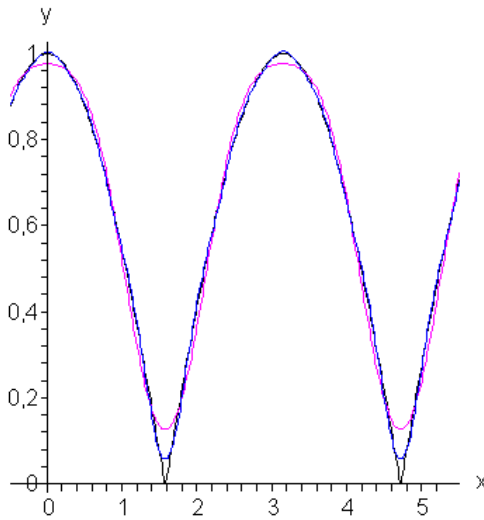


Figure 21. Fourier series of $f(x) = |\cos x|$, $n = 2$, $n = 5$.

Example 21.

Find the Fourier series of the function $f(x) = |\sin x|$.

Solution.

Since this function is even and $|\sin x| = \sin x$ for $x \in [0, \pi]$, we can write

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} \sin x dx = \frac{2}{\pi} \cdot (-\cos x) \Big|_0^{\pi} \\ &= \frac{2}{\pi} (-\cos \pi + \cos 0) = \frac{4}{\pi}. \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} \sin x \cos nx dx$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

$$\begin{aligned}
 &= \frac{1}{\pi} \int_0^{\pi} [\sin(nx+x) - \sin(nx-x)] dx \\
 &= \frac{1}{\pi} \int_0^{\pi} [\sin(n+1)x - \sin(n-1)x] dx \\
 &= -\frac{1}{\pi} \left[\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1} \right]_0^{\pi} \\
 &= -\frac{1}{\pi} \left[\frac{(-1)^{n+1} - 1}{n+1} - \frac{(-1)^{n+1} - 1}{n-1} \right] \\
 &= -\frac{1}{\pi} \left[-\frac{(-1)^n + 1}{n+1} + \frac{(-1)^n + 1}{n-1} \right] \\
 &= -\frac{(-1)^n + 1}{\pi} \cdot \frac{(-n+1+n+1)}{n^2-1} = -\frac{2}{\pi} \cdot \frac{(-1)^n + 1}{n^2-1}, \text{ if } n \neq 1.
 \end{aligned}$$

For case $n=1$, we have

$$\begin{aligned}
 a_1 &= \frac{2}{\pi} \int_0^{\pi} \sin x \cos x \, dx = \frac{1}{\pi} \int_0^{\pi} \sin 2x \, dx \\
 &= \frac{1}{\pi} \left(-\frac{\cos 2x}{2} \right) \Big|_0^{\pi} = \frac{1}{2\pi} (-\cos 2\pi + \cos 0) = 0.
 \end{aligned}$$

Since the coefficients $b_n = 0$, the Fourier series has the form

$$f(x) = |\sin x| = \frac{2}{\pi} - \frac{2}{\pi} \sum_{n=2}^{\infty} \frac{(-1)^n + 1}{n^2 - 1} \cos nx.$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

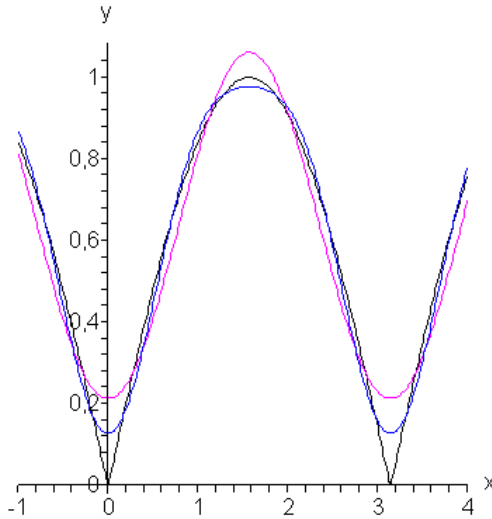


Figure 22. Fourier series of $f(x) = |\sin x|$, $n = 3$, $n = 5$.

Example 22.

Find the Fourier series of the function $f(x) = (x)$, where (x) denotes the distance from the nearest integer.

Solution.

This function is even on the interval $[0, 1]$ and has period 1

($L = \frac{1}{2}$). Then $b_n = 0$.

The constant a_0 is

$$a_0 = \frac{2}{L} \int_0^L f(x) dx = 4 \int_0^{\frac{1}{2}} x dx = 4 \cdot \left(\frac{x^2}{2} \right) \Big|_0^{\frac{1}{2}} = \frac{1}{2}.$$

Using integration by parts, we have

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = 4 \int_0^{\frac{1}{2}} x \cos 2n\pi x dx \\ &= 4 \left[\left(x \cdot \frac{\sin 2n\pi x}{2n\pi} \right) \Big|_0^{\frac{1}{2}} - \frac{1}{2n\pi} \int_0^{\frac{1}{2}} \sin 2n\pi x dx \right] \\ &= \frac{2}{n\pi} \left[\left(x \sin 2n\pi x \right) \Big|_0^{\frac{1}{2}} + \left(\frac{\cos 2n\pi x}{2n\pi} \right) \Big|_0^{\frac{1}{2}} \right] \\ &= \frac{2}{n\pi} \left[\frac{1}{2} \sin n\pi + \frac{1}{2n\pi} (\cos n\pi - 1) \right] \\ &= \frac{1}{n^2 \pi^2} [(-1)^n - 1]. \end{aligned}$$

For even $n = 2k$, $a_{2k} = 0$.

For odd $n = 2k + 1$, $a_{2k+1} = \frac{-2}{(2k+1)^2 \pi^2}$, $k = 0, 1, 2, 3, \dots$

Thus,

$$f(x) = \frac{1}{4} - \frac{2}{\pi^2} \sum_{k=0}^{\infty} \frac{\cos(4k+2)\pi x}{(2k+1)^2}.$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

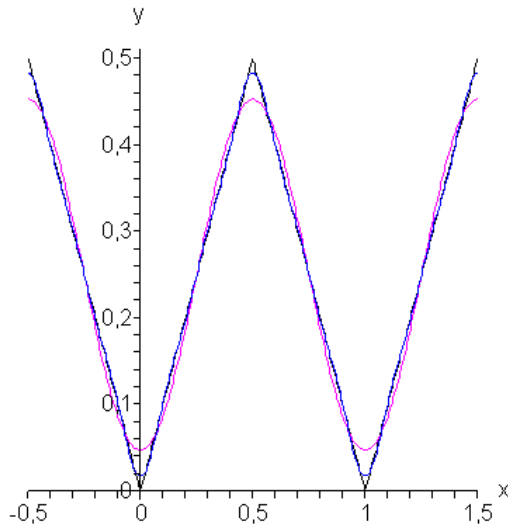


Figure 23. Fourier series of $f(x) = x$, $n = 0$, $n = 2$.

Example 23.

Find the Fourier series of the function

$$f(x) = \begin{cases} x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{if } 1 < x \leq 2 \\ 3 - x, & \text{if } 2 < x \leq 3 \end{cases}$$

(a trapezoidal wave).

Solution.

Here $L = \frac{3}{2}$. Then

$$a_0 = \frac{1}{L} \int_a^b f(x) dx = \frac{2}{3} \int_0^3 f(x) dx = \frac{2}{3} \left[\int_0^1 x dx + \int_1^2 dx + \int_2^3 (3-x) dx \right]$$

2. FOURIER SERIES OF FUNCTIONS WITH AN ARBITRARY PERIOD

$$= \frac{2}{3} \left[\left(\frac{x^2}{2} \right) \Big|_0^1 + x \Big|_1^2 + \left(3x - \frac{x^2}{2} \right) \Big|_2^3 \right] = \frac{4}{3}.$$

$$\begin{aligned} a_n &= \frac{1}{L} \int_a^b f(x) \cos \frac{n\pi x}{L} dx = \frac{2}{3} \int_0^3 f(x) \cos \frac{2n\pi x}{3} dx \\ &= \frac{2}{3} \left\{ \int_0^1 x \cos \frac{2n\pi x}{3} dx + \int_1^2 \cos \frac{2n\pi x}{3} dx + \int_2^3 (3-x) \cos \frac{2n\pi x}{3} dx \right\} \\ &= \frac{2}{3} \left\{ \left[\left(\frac{3}{2n\pi} x \sin \frac{2n\pi x}{3} \right) \Big|_0^1 - \int_0^1 \frac{3}{2n\pi} \sin \frac{2n\pi x}{3} dx \right] + \left(\frac{3}{2n\pi} \sin \frac{2n\pi x}{3} \right) \Big|_1^2 \right. \\ &\quad \left. + \left[\left(\frac{3}{2n\pi} (3-x) \sin \frac{2n\pi x}{3} \right) \Big|_2^3 + \int_2^3 \frac{3}{2n\pi} \sin \frac{2n\pi x}{3} dx \right] \right\} \\ &= \frac{2}{3} \left\{ \frac{3}{2n\pi} \sin \frac{2n\pi}{3} + \frac{9}{4n^2\pi^2} \left(\cos \frac{2n\pi}{3} - 1 \right) + \frac{3}{2n\pi} \left(\sin \frac{4n\pi}{3} - \sin \frac{2n\pi}{3} \right) \right. \\ &\quad \left. - \frac{3}{2n\pi} \sin \frac{4n\pi}{3} + \frac{9}{4n^2\pi^2} \left(-\cos 2n\pi + \cos \frac{4n\pi}{3} \right) \right\} \\ &= \frac{2}{3} \left\{ \frac{9}{4n^2\pi^2} \left(\cos \frac{2n\pi}{3} - 1 \right) + \frac{9}{4n^2\pi^2} \left(\cos \frac{4n\pi}{3} - 1 \right) \right\}. \end{aligned}$$

Since $\cos \frac{4n\pi}{3} = \cos \left(2n\pi - \frac{2n\pi}{3} \right) = \cos \frac{2n\pi}{3}$, we obtain

$$a_n = \frac{2}{3} \cdot \frac{9}{4n^2\pi^2} \left(\cos \frac{2n\pi}{3} - 1 \right) = \frac{3}{n^2\pi^2} \left(\cos \frac{2n\pi}{3} - 1 \right), \text{ where } n = 1, 2, 3, \dots$$

The coefficients $b_n = 0$ because the function is even on the given interval $[0, 3]$. Then

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$$f(x) = \frac{2}{3} - \frac{3}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - \cos \frac{2n\pi}{3}}{n^2} \cos \frac{2n\pi x}{3}.$$

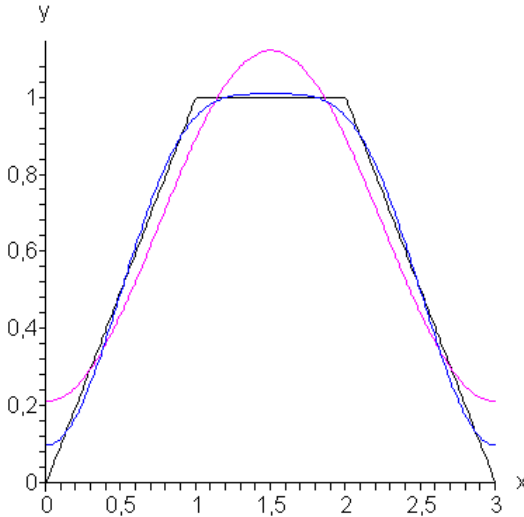


Figure 24. Fourier series for trapezoidal wave, $n = 1$, $n = 3$.

Example 24.

Find the Fourier series of the function $f(x) = x$, defined on the interval $[a, b]$.

Solution.

We set $L = \frac{b-a}{2}$. Supposing that the given $2L$ -periodic function can be represented by the convergent Fourier series, we can write

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right).$$

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Calculate the Fourier coefficients a_0 , a_n , b_n .

$$\begin{aligned} a_0 &= \frac{1}{L} \int_a^b f(x) dx = \frac{1}{L} \int_a^{a+2L} x dx = \frac{1}{L} \left(\frac{x^2}{2} \right) \Big|_a^{a+2L} = \frac{1}{2L} [(a+2L)^2 - a^2] \\ &= \frac{1}{2L} [a^2 + 4aL + 4L^2 - a^2] = \frac{4aL + 4L^2}{2L} = 2(a+L). \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned} a_n &= \frac{1}{L} \int_a^b f(x) \cos \frac{n\pi x}{L} dx = \frac{1}{L} \int_a^{a+2L} x \cos \frac{n\pi x}{L} dx \\ &= \left[\begin{array}{l} u = x \\ dv = \cos \frac{n\pi x}{L} dx \\ du = dx \\ v = \int \cos \frac{n\pi x}{L} dx = \frac{L}{n\pi} \sin \frac{n\pi x}{L} \end{array} \right] \\ &= \frac{1}{L} \left[\left(\frac{xL}{n\pi} \sin \frac{n\pi x}{L} \right) \Big|_a^{a+2L} - \int_a^{a+2L} \frac{L}{n\pi} \sin \frac{n\pi x}{L} dx \right] \\ &= \frac{1}{n\pi} \left[\left(x \sin \frac{n\pi x}{L} \right) \Big|_a^{a+2L} + \left(\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right) \Big|_a^{a+2L} \right] \\ &= \frac{1}{n\pi} \left[(a+2L) \sin \left(\frac{n\pi a}{L} + 2n\pi \right) - a \sin \frac{n\pi a}{L} + \right. \\ &\quad \left. + \frac{L}{n\pi} \cos \left(\frac{n\pi a}{L} + 2n\pi \right) - \frac{L}{n\pi} \cos \frac{n\pi a}{L} \right] \\ &= \frac{1}{n\pi} \left[(a+2L) \sin \frac{n\pi a}{L} - a \sin \frac{n\pi a}{L} + \frac{L}{n\pi} \cos \frac{n\pi a}{L} - \frac{L}{n\pi} \cos \frac{n\pi a}{L} \right] \\ &= \frac{2L}{n\pi} \sin \frac{n\pi a}{L}. \end{aligned}$$

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Similarly, we find

$$\begin{aligned}
 b_n &= \frac{1}{L} \int_a^b f(x) \sin \frac{n\pi x}{L} dx = \frac{1}{L} \int_a^{a+2L} x \sin \frac{n\pi x}{L} dx \\
 &= \left[\begin{array}{l} u = x \\ dv = \sin \frac{n\pi x}{L} dx \\ du = dx \\ v = \int \sin \frac{n\pi x}{L} dx = -\frac{L}{n\pi} \cos \frac{n\pi x}{L} \end{array} \right] \\
 &= \frac{1}{L} \left[\left(-\frac{xL}{n\pi} \cos \frac{n\pi x}{L} \right) \Big|_a^{a+2L} - \int_a^{a+2L} \left(-\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right) dx \right] \\
 &= -\frac{1}{n\pi} \left[\left(x \cos \frac{n\pi x}{L} \right) \Big|_a^{a+2L} - \left(\frac{L}{n\pi} \sin \frac{n\pi x}{L} \right) \Big|_a^{a+2L} \right] \\
 &= -\frac{1}{n\pi} \left[(a+2L) \cos \left(\frac{n\pi a}{L} + 2n\pi \right) - a \cos \frac{n\pi a}{L} - \right. \\
 &\quad \left. - \frac{L}{n\pi} \sin \left(\frac{n\pi a}{L} + 2n\pi \right) + \frac{L}{n\pi} \sin \frac{n\pi a}{L} \right] \\
 &= -\frac{1}{n\pi} \left[(a+2L) \cos \frac{n\pi a}{L} - a \cos \frac{n\pi a}{L} - \frac{L}{n\pi} \sin \frac{n\pi a}{L} + \frac{L}{n\pi} \sin \frac{n\pi a}{L} \right] \\
 &= -\frac{2L}{n\pi} \cos \frac{n\pi a}{L}.
 \end{aligned}$$

Thus, the Fourier series representation of the function is

$$\begin{aligned}
 f(x) &= a + L + \frac{2L}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin \frac{n\pi a}{L} \cos \frac{n\pi x}{L} - \frac{1}{n} \cos \frac{n\pi a}{L} \sin \frac{n\pi x}{L} \right) \\
 &= a + L + \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi(a-x)}{L}.
 \end{aligned}$$

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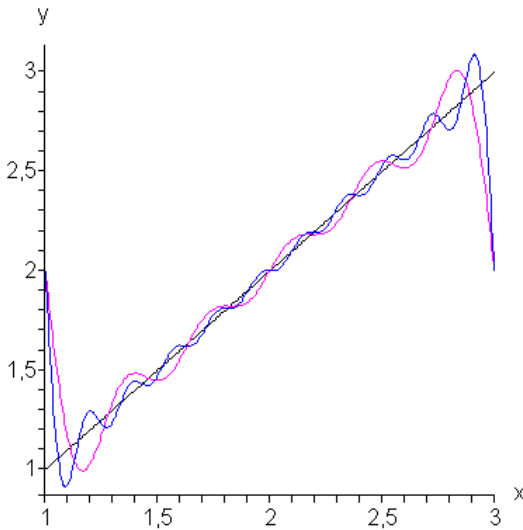


Figure 25. Fourier series of $f(x) = x$, $n = 5$, $n = 10$.

Example 25.

Find the Fourier series of the function $f(x) = \cos^2 x$.

Solution.

This function is even with period π ($L = \frac{\pi}{2}$). Therefore, the coefficients $b_n = 0$.

Compute a_0 and a_n .

$$\begin{aligned} a_0 &= \frac{2}{L} \int_0^L f(x) dx = \frac{4}{\pi} \int_0^{\pi/2} \cos^2 x dx = \frac{2}{\pi} \int_0^{\pi/2} (1 + \cos 2x) dx \\ &= \frac{2}{\pi} \left[x + \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{2}{\pi} \left[\frac{\pi}{2} + \frac{\sin \pi}{2} \right] = 1. \end{aligned}$$

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$$\begin{aligned}
 a_n &= \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx = \frac{4}{\pi} \int_0^{\pi/2} \cos^2 x \cos 2nx dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} (1 + \cos 2x) \cos 2nx dx = \frac{2}{\pi} \int_0^{\pi/2} (\cos 2nx + \cos 2x \cos 2nx) dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} \left\{ \cos 2nx + \frac{1}{2} [\cos(2n-2)x + \cos(2n+2)x] \right\} dx \\
 &= \frac{1}{\pi} \int_0^{\pi/2} [2 \cos 2nx + \cos(2n-2)x + \cos(2n+2)x] dx \\
 &= \frac{1}{\pi} \left[\frac{\sin 2nx}{n} + \frac{\sin(2n-2)x}{2n-2} + \frac{\sin(2n+2)x}{2n+2} \right]_0^{\pi/2} \\
 &= \frac{1}{\pi} \left[\frac{\sin n\pi}{n} + \frac{\sin(n-1)\pi}{2n-2} + \frac{\sin(n+1)\pi}{2n+2} \right] = 0.
 \end{aligned}$$

However, this result is valid only for $n \geq 2$. Therefore we calculate coefficient a_1 separately.

$$\begin{aligned}
 a_1 &= \frac{4}{\pi} \int_0^{\pi/2} \cos^2 x \cos 2x dx = \frac{2}{\pi} \int_0^{\pi/2} (1 + \cos 2x) \cos 2x dx \\
 &= \frac{2}{\pi} \int_0^{\pi/2} (\cos 2x + \cos^2 2x) dx = \frac{2}{\pi} \int_0^{\pi/2} \left(\cos 2x + \frac{1 + \cos 4x}{2} \right) dx \\
 &= \frac{1}{\pi} \int_0^{\pi/2} (2 \cos 2x + 1 + \cos 4x) dx = \frac{1}{\pi} \left(\sin 2x + x + \frac{\sin 4x}{4} \right) \Big|_0^{\pi/2} \\
 &= \frac{1}{\pi} \left(\sin \pi + \frac{\pi}{2} + \frac{\sin 2\pi}{4} \right) = \frac{1}{2}.
 \end{aligned}$$

Thus, the Fourier series of the function $\cos^2 x$ is given by

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$$f(x) = \cos^2 x = \frac{1}{2} + \frac{1}{2} \cos 2x .$$

This result is the well-known trigonometric identity.